A NOTE ON PYTHAGOGEAN TRIPLETS

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A Pythagorean triplet is defined as a, b, c, in which $a^2 + b^2 = c^2$. It is well known that, where u and v are any two integers, $a = u^2 - v^2$, $b = 2uv$, and $c = u^2 + v^2$.

Triplets like 9, 40, 41, and 133, 156, 205, are of particular interest because $a + b$ is also a square. Not all Pythagorean triplets possess this property; for example, 3, 4, 5, and 20, 21, 29.

I have found that, x and y being any two integers, Pythagorean triplets possessing this property can be generated where $u = x^2 + (x + y)^2$ and $v = 2y(x + y)$. Then

I. $$a = u^2 - v^2 = 4x^4 + 8x^3y + 4x^2y^2 - 4xy^3 - 3y^4$$

II. $$b = 2uv = 8x^3y + 16x^2y^2 + 12xy^3 + 4y^4$$

III. $$c = u^2 + v^2 = 4x^4 + 8x^3y + 12x^2y^2 + 12xy^3 + 5y^4$$

IV. $$a + b = (2x^2 + 4xy + y^2)^2$$

V. $$b + c = (2x^2 + 4xy + 3y^2)^2 .$$

In triplets like 3, 4, 5, and 5, 12, 13, where $u = v + 1$, there is the further property that $a^2 = b + c$. Of the triplets in the series in which $a^2 = b + c$, only certain triplets possess the property that $a + b$ is also a square. The first six such triplets are listed below:

u	v	a	b	c
5	4	9	40	41
29	28	57	1,624	1,625
169	168	337	56,784	56,785

985	984	1,969	1,938,480	1,938,481
5,741	5,740	11,481	65,906,680	65,906,681
33,461	33,460	66,921	2,239,210,120	2,239,210,121

The series of u's (5, 29, 169, 985, $\cdots$) is a recurrent series which is defined as

$$u_n = 6u_{n-1} - u_{n-2} ,$$

where $u_0 = 1$ and $u_1 = 5$.

Since the generator

$$u = x^2 + (x + y)^2 ,$$

it can be expressed as the sum of two squares:

$$\begin{aligned}
u_1 &= 1^2 + 2^2 = 5 \\
u_2 &= 2^2 + 5^2 = 29 \\
u_3 &= 5^2 + 12^2 = 169 \\
u_4 &= 12^2 + 29^2 = 985 \\
u_5 &= 29^2 + 70^2 = 5741 \\
u_6 &= 70^2 + 169^2 = 33,461 \\
&\vdots
\end{aligned}$$

As expressed in this manner, the series of u's forms the recurrent series

$$\begin{aligned}
u_1 &= 1^2 + 2^2 = 5 \\
u_2 &= 2^2 + (1 + 2\cdot 2)^2 = 29 \\
u_3 &= 5^2 + (2 + 2\cdot 5)^2 = 169 \\
u_4 &= 12^2 + (5 + 2\cdot 12)^2 = 985 \\
u_5 &= 29^2 + (12 + 2\cdot 29)^2 = 5741 \\
u_6 &= 70^2 + (29 + 2\cdot 70)^2 = 33,461 \\
&\vdots
\end{aligned}$$

Pythagorean triplets possessing the properties that (1) $a^2 = b + c$ and that (2) $a + b$ is a square can be derived in another way.

For a triplet to possess the first property, the necessary and sufficient condition is that $u = v + 1$:

$$(u^2 - v^2)^2 = 2uv + u^2 + v^2$$

$$(u^2 - v^2)^2 = (u + v)^2$$

$$u^2 - v^2 = u + v$$

$$(u - v)(u + v) = u + v$$

$$u - v = 1$$

$$u = v + 1 .$$

We already know that for a triplet to possess property (2),

$$u = x^2 + (x + y)^2$$

and

$$v = 2y(x + y) .$$

Since $u = v + 1$, set

$$x^2 + (x + y)^2 = 2y(x + y) + 1 .$$

Then

$$x = \pm\sqrt{\frac{y^2 + 1}{2}}$$

(symbolized by 1) and

$$y = \pm\sqrt{2x^2 - 1}$$

(symbolized by k).

Substituting

$$x = \pm\sqrt{\frac{y^2 + 1}{2}}$$

in Eqs. I, II, III, IV, and V, we find that

$$a = 4y^2 + 4yl + 1$$

$$b = 12y^4 + 16y^3l + 8y^2 + 4yl$$

$$c = b + 1$$

$$a + b = (2y^2 + 4yl + 1)^2$$

$$b + c = (4y^2 + 4yl + 1)^2$$

Now

$$\pm\sqrt{\frac{y^2 + 1}{2}}$$

is integral for 1, 7, 41, 239, $\cdots$. This is a recurrent series which is defined as

$$r_n = 6r_{n-1} - r_{n-2} ,$$

where $r_1 = 1$ and $r_2 = 7$. Substituting alternately the positive and negative values of

$$\pm\sqrt{\frac{y^2 + 1}{2}}$$

in a, b, c, we obtain the desired triplets.

Substituting $y = \pm\sqrt{2x^2 - 1}$ in Eqs. I, II, III, IV, and V, we find that

[Continued on page 212.]