NOTE ON THE CHARACTERISTIC NUMBER OF A SEQUENCE OF FIBONACCI SQUARES

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Given a sequence of squares formed from the terms of a general Fibonacci sequence. It is proposed to set up a quadratic expression that will characterize a given sequence of this type.

First let it be noted that since this is equivalent to an expression of the fourth degree in Fibonacci numbers, the characteristic number would be a constant that would not oscillate in sign. To find such an expression we may proceed as follows.

Let the original sequence be given by $H_n = Ar^n + Bs^n$ where r and s are the roots of the Fibonacci recursion relation. Then the square term

$$G_n = H_n^2 = A^2r^{2n} + 2AB(rs)^n + B^2s^{2n} .$$

We now calculate three expressions.

$$\begin{aligned} G_n^2 &= A^4r^{4n} + 6A^2B^2 + B^4s^{4n} + 4A^3B(rs)^n r^{2n} + 4AB^3(rs)s^{2n} \\ G_{n-1}G_{n+1} &= A^4r^{4n} + 4A^2B^2 + B^4s^{4n} + 2A^3B[r^{2n-2}(rs)^{n+1} + (rs)^{n-1}r^{2n+2}] \\ &\quad + 2AB^3[(rs)^{n-1}s^{2n+2} + (rs)^{n+1}s^{2n-2}] \\ &\quad + A^2B^2[r^{2n-2}s^{2n+2} + r^{2n+2}s^{2n-2}] \\ G_{n-2}G_{n+2} &= A^4r^{4n} + 4A^2B^2 + B^4s^{4n} + 2AB[r^{2n-4}(rs)^{n+2} + r^{2n+4}(rs)^{n-2}] \\ &\quad + 2AB^3[s^{2n+4}(rs)^{n-2} + s^{2n-4}(rs)^{n+2}] \\ &\quad + A^2B^2[r^{2n-4}s^{2n+4} + r^{2n+4}s^{2n-4}] . \end{aligned}$$

First let it be noted that the A^2B^2 terms which end the expressions for G_{n-1}, G_{n+1} and $G_{n-2}G_{n+2}$ are $7A^2B^2$ and $47A^2B^2$, respectively. The AB^3 and A^3B terms of $G_{n-1}G_{n+1}$ can be written together as

$$2AB(-1)^{n-1}[A^2r^{2n-2} + B^2s^{2n-2}] + 2AB(-1)^{n-1}[A^2r^{2n+2} + B^2s^{2n+2}] .$$

A similar expression can be obtained for the corresponding terms of G_{n-2}, G_{n+2}. If we let $G_{2n}^* = A^2r^{2n} + B^2s^{2n}$ we have the following relations.

$$G_n^2 = A^4r^{4n} + B^4s^{4n} + 6A^2B^2 + 4AB(-1)^nG_{2n}^*$$

$$G_{n-1}G_{n+1} = A^4r^{4n} + B^4s^{4n} + 11A^2B^2 + 6AB(-1)^{n-1}G^*_{2n}$$
$$G_{n-2}G_{n+2} = A^4r^{4n} + B^4s^{4n} + 51A^2B^2 + 14AB(-1)^n G^*_{2n} .$$

To eliminate all but the terms in A^2B^2 we need three multipliers x, y, z satisfying the relations

$$x + y + z = 0$$
$$-4x + 6y - 14z = 0$$

with the solution $x:y:x = -20:10:10$. Hence the required expression which gives a characteristic number of a quadratic character is

$$2G_n^2 - G_{n-1}G_{n+1} - G_{n-2}G_{n+2} = k .$$

The value of this expression is $K = -50A^2B^2 = -2D^2$ since the characteristic number of the original Fibonacci sequence is given by $D = 5AB$ where D is defined as $H_2^2 - H_1H_3$.

If the initial terms of the sequence of squares are a, b, c, the next two terms are given by the recursion relation $T_{n+1} = 2T_n + 2T_{n-1} - T_{n-2}$. Hence the fourth and fifth terms are $2c + 2b - a$ and $-2a + 3b + 6c$. We form K from these beginning terms of the sequence and find an expression

$$K = 2a^2 - 2b^2 + 2c^2 - 2ab - 2bc - 6ac .$$

a, b, and c are related by the relation $\sqrt{c} = \sqrt{a} + \sqrt{b}$ which becomes

$$a^2 + b^2 + c^2 - 2ab - 2bc - 2ca = 0.$$

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