ELEMENTARY PROBLEMS AND SOLUTIONS

Edited by
A. P. HILLMAN
University of New Mexico, Albuquerque, New Mexico

Send all communications regarding Elementary Problems and Solutions to Professor A. P. Hillman, Dept. of Mathematics and Statistics, University of New Mexico, Albuquerque, New Mexico 87106. Each problem or solution should be submitted in legible form, preferably typed in double spacing, on a separate sheet or sheets, in the format used below. Solutions should be received within three months of the publication date.

Contributors (in the United States) who desire acknowledgement of receipt of their contributions are asked to enclose self-addressed postcards.

DEFINITIONS

$F_0 = 0, \quad F_1 = 1, \quad F_{n+2} = F_{n+1} + F_n; \quad L_0 = 2, \quad L_1 = 1, \quad L_{n+2} = L_{n+1} + L_n.$

PROBLEMS PROPOSED IN THIS ISSUE

B-232 Proposed by Guy A. R. Guillotte, Quebec, Canada.

In the following multiplication alphametic, the five letters, F, Q, I, N, and E represent distinct digits. The dashes denote not necessarily distinct digits. What are the digits of FINE FQ ?

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  FQ
  FQ
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FINE
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B-233 Proposed by Harlan L. Umansky, Emerson High School, Union City, N. J.

Show that the roots of

$$F_{n-1}x^2 - F_nx - F_{n+1} = 0$$

are $x = -1$ and $x = F_{n+1}/F_{n-1}$. Generalize to show a similar result for all sequences formed in the same manner as the Fibonacci sequence.

B-234 Proposed by W. C. Barley, Los Gatos High School, Los Gatos, California

Prove that

$$L_n^3 = 2F_{n-1}^3 + F_n^3 + 6F_{n-1}F_{n+1}^2 .$$

B-235 Proposed by Phil Mana, University of New Mexico, Albuquerque, New Mexico.

Find the largest positive integer n such that F_n is smaller than the sum of the cubes of the digits of F_n.

B-236 Proposed by Paul S. Bruckman, San Rafael, California.

Let P_n denote the probability that, in n throws of a coin, two consecutive heads will not appear. Prove that

$$P_n = 2^{-n}F_{n+2} .$$

B-237 Proposed by V. E. Hoggatt, Jr., San Jose State College, San Jose, California.

Let (m,n) denote the greatest common divisor of the integers m and n.

(i) Given $(a,b) = 1$, prove that $(a^2 + b^2,\ a^2 + 2ab)$ is 1 or 5.

(ii) Prove the converse of Part (i).

APOLOGIES FOR SOME OMISSIONS

Following are some of the solvers whose names were inadvertently omitted from the lists of solvers of previous problems:

B-197 David Zeitlin

B-202 Herta T. Freitag, N. J. Kuenzi and Robert W. Prielipp

B-203 Herta T. Freitag and Robert W. Prielipp

B-206 Herta T. Freitag

B-207 Herta T. Freitag

SOLUTIONS

LUCKY 11 MODULO UNLUCKY 13

B-214 Proposed by R. M. Grassl, University of New Mexico, Albuquerque, New Mexico.

Let n be a random positive integer. What is the probability that L_n has a remainder of 11 on division by 13? [Hint: Look at the remainders for n = 1, 2, 3, 4, 5, 6, ⋯.]

Composite of solutions by Paul S. Bruckman, San Rafael, California, and Phil Mana, Albuquerque, New Mexico.

Let R_n be the remainder in the division of L_n by 13. Then

$$R_{n+2} \equiv R_{n+1} + R_n \pmod{13}.$$

Calculating the first 30 values of R_n, one finds that $R_{29} = 1 = R_1$ and $R_{30} = 3 = R_2$. It then follows from the recursion formula that $R_{n+28} = R_n$. The only n's with $R_n = 11$ and $1 \le n \le 28$ are n = 5, 9, and 14. Hence, in each cycle of 28 terms, the remainder 11 occurs exactly 3 times. Therefore, the required probability is 3/28.

Also solved by Debby Hesse and the Proposer.

QUOTIENT OF POLYNOMIALS

B-215 Proposed by Phil Mana, University of New Mexico, Albuquerque, New Mexico.

Prove that for all positive integers n the quadratic $q(x) = x^2 - x - 1$ is an exact divisor of the polynomial

$$P_n(x) = x^{2n} - L_n x^n + (-1)^n$$

and establish the nature of $p_n(x)/q(x)$. [Hint: Evaluate $p_n(x)/q(x)$ for n = 1, 2, 3, 4, 5.]

Solution by L. Carlitz, Duke University, Durham, North Carolina.

Let α, β denote the roots of $x^2 - x - 1$. Since $L_n = \alpha^n + \beta^n$, it is clear that

$$Q_n(x) = \frac{x^{2n} - L_n x^n + (-1)}{x^2 - x - 1} = \frac{(x^n - \alpha^n)(x^n - \beta^n)}{(x - \alpha)(x - \beta)}$$

is a polynomial.

To find the coefficients of $Q_n(x)$ we put

$$Q_n(x) = \sum_{r=0}^{n-1} \alpha^r x^{n-r-1} \sum_{s=0}^{n-1} \beta^s x^{n-s-1}$$

$$= \sum_{k=0}^{2n-2} \left(x^{2n-k-2} \sum_{\substack{r+s=k \\ r<n,\, s<n}} \alpha^r \beta^s \right) = \sum_{k=0}^{2n-2} x^{2n-k-2} c_k ,$$

say. Then for $k \leq n - 1$,

$$c_k = \sum_{r+s=k} \alpha^r \beta^s = \frac{\alpha^{k+1} - \beta^{k+1}}{\alpha - \beta} = F_{k+1} .$$

For $k \geq n$, we have

$$c_k = \sum_{r=k-n+1}^{n-1} \alpha^r \beta^{k-r} = (\alpha\beta)^{k-n+1} \sum_{j=0}^{2n-k-2} \alpha^j \beta^{2n-k-j-2}$$

$$= (-1)^{k-n+1} F_{2n-k-1} .$$

Also solved by Paul S. Bruckman, Ralph Garfield, G. A. R. Guillotte, Herta T. Freitag, David Zeitlin, and the Proposer.

A NONHOMOGENEOUS RECURSION

B-216 Proposed by V. E. Hoggatt, Jr., San Jose State College, San Jose, California.

Solve the recurrence $D_{n+1} = D_n + L_{2n} - 1$ for D_n, subject to the initial condition $D_1 = 1$.

Solution by David Zeitlin, Minneapolis, Minnesota.

Since $D_0 = 0$ and

$$\sum_{k=0}^{n-1} L_{2k} = 1 + L_{2n-1} ,$$

we have, with n replaced by k in the recurrence,

$$D_n = \sum_{k=0}^{n-1} (D_{k+1} - D_k) = \sum_{k=0}^{n-1} (-1) + \sum_{k=0}^{n-1} L_{2k}$$

$$= -n + 1 + L_{2n-1} .$$

Also solved by Paul S. Bruckman, Herta T. Freitag, Ralph Garfield, G. A. R. Guillotte, and the Proposer.

MODIFIED PASCAL TRIANGLE

B-217 Proposed by L. Carlitz, Duke University, Durham, North Carolina.

A triangular array of numbers $A(n,k)$ $(n = 0, 1, 2, \cdots, 0 \le k \le n)$ is defined by the recurrence

$$A(n + 1,k) = A(n, k - 1) + (n + k + 1)A(n,k) \qquad (1 \le k \le n)$$

together with the boundary conditions

$$A(n,0) = n! , \quad A(n,n) = 1 .$$

Find an explicit formula for $A(n,k)$.

Solution by Paul S. Bruckman.

Let $A(n,k) = (n!/k!)B(n,k)$. Substituting this expression in the given recursion, we obtain

$$[(n+1)!/k!]B(n+1,k) = [n!/(k-1)!]B(n, k-1) + [(n+k+1)n!/k!]B(n,k) .$$

Multiplying throughout by $k!/n!$ gives us

$$(n+1)B(n+1, k) = kB(n, k-1) + (n+k+1)B(n,k)$$

or

$$(1) \qquad (n+1)[B(n+1, k) - B(n,k)] = k[B(n, k-1) + B(n,k)] .$$

Next we demonstrate that recursion (1) is satisfied by

$$B(n, k) = \binom{n}{k} .$$

Since

$$\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1} \quad \text{and} \quad \binom{m-1}{r-1} = r\binom{m}{r},$$

$$(n+1)\left[\binom{n+1}{k} - \binom{n}{k}\right] = (n+1)\binom{n}{k-1} = k\binom{n+1}{k}.$$

Also,

$$k\left[\binom{n}{k-1} + \binom{n}{k}\right] = k\binom{n+1}{k} \quad .$$

Therefore,

$$B(n, k) = \binom{n}{k}$$

satisfies (1). The boundary conditions for this $B(n,k)$ are $B(n,0) = 1 = B(n,n)$, which lead to the desired boundary conditions for

$$A(n,k) = (n!/k!)B(n,k) = \frac{n!}{k!}\binom{n}{k} = (n-k)!\binom{n}{k}^2 = \frac{(n!)^2}{(n-k)!(k!)^2} .$$

Also solved by David Zeitlin and the Proposer.

ARCTAN OF A SUM EQUALS SUM OF ARCTANS

B-218 Proposed by Guy A. R. Guillotte, Montreal, Quebec, Canada.

Let $a = (1 + \sqrt{5})/2$ and show that

$$\operatorname{Arctan} \sum_{n=1}^{\infty} [1/(aF_{n+1} + F_n)] = \sum_{n=1}^{\infty} \operatorname{Arctan} (1/F_{2n+1}) .$$

Solution by L. Carlitz, Duke University, Durham, North Carolina.

Since $aF_{n+1} + F_n = a^{n+1}$ and

$$\sum_{n=1}^{\infty} \frac{1}{a^{n+1}} = \frac{1}{a(a-1)} = 1 ,$$

the stated result may be replaced by

$$(*) \qquad \frac{\pi}{4} = \sum_{n=1}^{\infty} \arctan \frac{1}{F_{2n+1}} .$$

Now

$$\arctan \frac{1}{F_{2n}} - \arctan \frac{1}{F_{2n+1}} = \arctan \frac{\dfrac{1}{F_{2n}} - \dfrac{1}{F_{2n+1}}}{1 + \dfrac{1}{F_{2n}F_{2n+1}}}$$

$$= \arctan \frac{F_{2n-1}}{F_{2n}F_{2n+1} + 1} = \arctan \frac{1}{F_{2n+2}} ,$$

using the well-known identity $F_{2n-1}F_{2n+2} - F_{2n}F_{2n+1} = 1$. Hence

$$\arctan \frac{1}{F_{2n+1}} = \arctan \frac{1}{F_{2n}} - \arctan \frac{1}{F_{2n+2}} .$$

Take $n = k,\ k+1,\ k+2, \cdots$ and add the resulting equations. We get

$$\sum_{n=k}^{\infty} \arctan \frac{1}{F_{2n+1}} = \arctan \frac{1}{F_{2k}} .$$

In particular, for $k = 1$, this reduces to (*).

Also solved by Paul S. Bruckman, David Zeitlin, and the Proposer.

HILBERT MATRIX

B-219 Proposed by Tomas Djerverson, Albrook College, Tigertown, New Mexico.

Let k be a fixed positive integer and let $a_0, a_1, \cdots, a_k$ be fixed real numbers such that, for all positive integers n.

$$\frac{a_0}{n} + \frac{a_1}{n+1} + \cdots + \frac{a_k}{n+k} = 0 .$$

Prove that $a_0 = a_1 = \cdots = a_k = 0$.

Solution by David Zeitlin, Minneapolis, Minnesota.

For $n = 1, 2, \cdots, k+1$, we have a homogeneous system of $(k+1)$ linear equations in the $k+1$ unknowns: $a_0, a_1, \cdots, a_k$. The coefficient matrix is the well-known Hilbert matrix, which is non-singular. Thus, the determinant of the system is non-zero; and so, by Cramer's rule, $a_0 = a_1 = a_2 = \cdots = a_k = 0$.

Also solved by Paul S. Bruckman and the Proposer.