A LUCAS NUMBER COUNTING PROBLEM

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Marshall Hall, Jr., [1] proposes the problem: Given

$$S_1, S_2, \cdots, S_n, \qquad S_i = (i, i+1, i+2)$$

(reduced mod 7, representing 0 as 7), show that there are 31 different sets, formed by choosing exactly one element from each original set and including each number from 1 to 7 exactly once.

The problem of how many new sets can be formed from this type of group of sets can be generalized in terms of the Fibonacci and Lucas numbers.

Given sets

$$S_1, S_2, \cdots, S_n, \qquad S_i = (i, i+1, i+2)$$

(reduced mod n, representing 0 as n), the number of new sets (for all $n \geq 4$) formed by choosing one element from each original set, including each number from 1 to n exactly once is $L_n + 2$, where L_n is the n^{th} Lucas number,

$$L_1 = 1, \quad L_2 = 3, \quad L_n = L_{n-1} + L_{n-2}.$$

$$F_1 = 1, \quad F_2 = 1, \quad F_n = F_{n-1} + F_{n-2}.$$

$$L_n = F_{n-1} + F_{n+1}.$$

One more Fibonacci identity is needed:

$$1 + 1 + 1 + 2 + 3 + 5 + \cdots + F_{n-3} = F_{n-1}.$$

The number of sets shall be counted by arranging the sets in ascending order (base $n+1$) to avoid missing any possible sets. A series in a group of sets

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which begin with the same number and are not determined by the first two numbers. The base set of a series is the set with all elements after the first arranged in ascending order; e.g., $\{2, 3, 4, 5, 6, 7, 8, 9, 1\}$.

The set beginning with 1, 2, is obviously determined.

The first base set is $\{1, 3, 4, 5, \cdots, n, 2\}$.

The 2 at the end of the set can't be chosen from any other set, so the first change which can be made must be the interchange of the n and the n - 1. There can be no other sets between them because there are no numbers less than n - 1 in the last two original sets; e.g., 1, 3, 4, 5, 6, 7, 8, 2 is changed to: 1, 3, 4, 5, 6, 8, 7, 2.

The interchange of n - 1 and n - 2 would create one new set.

The interchange of n - 2 and n - 3 would create two new sets (isomorphic to the first two of the series, but with the n - 2 and n - 3 reversed).

For the set $\{2, 3, 4, 5, 6, 7, 8, 1\}$, the first 3 interchanges create the sets

$$\{2, 3, 4, 5, 6, 7, 8, 1\}$$
$$\{2, 3, 4, 5, 6, 7, 1, 8\}$$
$$\{2, 3, 4, 5, 6, 8, 7, 1\}$$
$$\{2, 3, 4, 5, 7, 6, 8, 1\}$$
$$\{2, 3, 4, 5, 7, 6, 1, 8\}$$

(The last 2 sets are similar to the first 2 except for the interchange of 7 and 6.)

The new sets created by the 5^{th} interchange are:

$$\{2, 3, 5, 4, 6, 7, 8, 1\}$$
$$\{2, 3, 5, 4, 6, 7, 1, 8\}$$
$$\{2, 3, 5, 4, 6, 8, 7, 1\}$$
$$\{2, 3, 5, 4, 7, 6, 8, 1\}$$
$$\{2, 3, 5, 4, 7, ,6, 1, 8\}$$

The sets are similar to those created by the first 3 interchanges but with the 5 and 4 interchanged.

An interchange involves only two elements. All elements after those two are left unchanged. Therefore the i^{th} interchange creates as many new sets as all the interchanges before $i - 1$ did.

The $i - 1$ interchange creates as many new sets as all interchanges before $i - 2$. The number of new sets before the $i - 2$ interchange plus the number of sets created by the $i - 2$ interchange equals the number of sets created by the $i - 1$ interchange. Therefore the number of sets created by the i interchange is equal to the number of sets created by the $i - 1$ interchange plus the number of sets created by the $i - 2$ interchange.

There are $n - 3$ interchanges in the first series.

There are $n - 2$ interchanges in the second series because the position of the 1 is not determined.

The set beginning with 3, 4, is determined.

There are $n - 3$ interchanges in the third series because the interchange of 2 and 4 would produce a determined set.

The number of sets in the first series is:

$$1 + 1 + 1 + 2 + 3 + 5 + \cdots + F_{n-3} = F_{n-1} .$$

The number of sets in the second series is:

$$1 + 1 + 1 + 2 + 3 + 5 + \cdots + F_{n-2} = F_n .$$

The number of sets in the third series is:

$$1 + 1 + 1 + 2 + 3 + 5 + \cdots + F_{n-3} = F_{n-1} .$$

The number of determined sets is 2.

The sum is:

$$\begin{aligned} &F_{n-1} + F_n + F_{n-1} + 2 \\ &= F_{n+1} + F_{n-1} + 2 \\ &= L_n + 2 . \end{aligned}$$

REFERENCES

1. Marshall Hall, Combinatorial Theory, Blaisdell Publishing Company, Waltham, Mass., 1967 (Problem 1, p. 53).

CORRIGENDA
FOR
ON PARTLY ORDERED PARTITIONS OF A POSITIVE INTEGER
Appearing in the Fibonacci Quarterly, May, 1971

Lines 3 and 4 of Proof of Theorem 1, page 330, should read:

"Then each V'_j $(j \neq 1)$ which has the same components as V'_i (in a different order) will give the same partition of n as V_i after rearrangement, hence, $\cdots$."

The last line of page 330 should read: $(1 \le r \le n - 1)$.

The third line of the Proof of Theorem 3, page 331, should read:

$$U = (u_1, u_2, \cdots, u_r) \in [U] \quad .$$

On page 331, the second line of expression for $\phi_k(n)$ should read:

$$" = \left(1 + \sum_{r=1}^{n-2} \phi_{k-1}(r)\right) + \phi_{k-1}(n - 1) \quad ,$$
$$= \cdots \qquad "$$

In Table 1, page 332, values for ϕ_1 are:

$$\phi_1 \cdot 1, 2, 4, 7, 12, 19, 30, \cdots \qquad .$$

The first two lines of the Proof for Lemma 1, page 333, should read:

"Proof.

$$\sum_{r=0}^{n-j-1} \binom{j - 3 + r}{r} = \cdots$$
$$= \binom{n - 3}{n - j - 1} - 1 + 1$$
$$= \cdots \qquad ."$$

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