PROPERTIES OF TRIBONACCI NUMBERS

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1. INTRODUCTION

Let us define a sequence of Tribonacci numbers

(1.1) $$\{T_n\}_0^\infty = \{T_n(b,\ c,\ d;\ P,\ Q,\ R)\}_0^\infty$$

by

(1.2) $$T_n = bT_{n-1} + cT_{n-2} + dT_{n-3}\ ,$$

where n denotes an integer ≥ 3 and T_0, T_1, T_2 are the initial terms P, Q, R respectively. Then it is easy to show that the n^{th} term of this sequence is given by

(1.3) $$T_n = la^n + mb^n + nr^n\ ,$$

where a, b, r are the roots of $x^3 - bx^2 - cx - d = 0$ and 1, m, n satisfy the following system of equations, viz.,

(1.4) $$1 + m + n = P,\quad la + mb + ar = Q,\quad la^2 + mb^2 + nr^2 = K.$$

Our aim is to study the properties of this sequence. The 9 special forms which we will refer are as follows:

(i) $$\{T_n^{(1)}\}_0^\infty = \{T_n(b,\ c,\ d;\ 0,\ 1,\ b)\}_0^\infty\ ,$$

(ii) $$\{T_n^{(2)}\}_0^\infty = \{T_n(b,\ c,\ d;\ 1,\ 0,\ c)\}_0^\infty\ ,$$

(iii) $$\{T_n^{(3)}\}_0^\infty = \{T_n(b,\ c,\ d;\ 0,\ d,\ bd)\}_0^\infty\ ,$$

(iv) $$\{T_n^{(4)}\}_0^\infty = \{T_n(b,\ c,\ d;\ 0,\ 0,\ 1)\}_0^\infty\ ,$$

(v) $$\{T_n^{(5)}\}_0^\infty = \{T_n(b, c, d; 0, 1, 0)\}_0^\infty ,$$

(vi) $$\{T_n^{(6)}\}_0^\infty = \{T_n(b, c, d; 1, 0, 0)\}_0^\infty ,$$

(vii) $$\{T_n^{(7)}\}_0^\infty = \{T_n(b, c, d; 3, b, b^2 + 2c)\}_0^\infty ,$$

(viii) $$\{T_n^{(8)}\}_0^\infty = \{T_n(1, 1, 1; 0, 1, 0)\}_0^\infty ,$$

(ix) $$\{T_n^{(9)}\}_0^\infty = \{T_n(1, 1, 1; 0, 0, 1)\}_0^\infty .$$

2. PROPERTIES OF $\{T_n\}_0^\infty$

First, we recall the following useful relations, viz.,

(2.1)
$$\begin{aligned} l &= [\{R - Q(b - a) + Pd/a\}(b - r)] \div D, \\ m &= [\{R - Q(b - b) + Pd/b\}(r - n)] \div D, \\ n &= [\{R - Q(b - r) + Pd/r\}(a - b)] \div D, \end{aligned}$$

where $D = (a - b)(b - r)(a - r)$;

(2.2) $$a = a' + b/3, \quad b = b' + b/3, \quad r = r' + b/3 ,$$

where a', b', r' are the roots of the reduced cubic equations $z^3 + 3Hz + G = 0$;

(2.3) $$a' = A^{1/3} + B^{1/3}, \quad b' = wA^{1/3} + w^2B^{1/3}, \quad r' = w^2A^{1/3} + wB^{1/3},$$

where $A, B = \{-G \pm \sqrt{(G^2 + 4H^3)}\}/2$;

(2.4) $$D - D' = 3(w - w^2)\sqrt{(G^2 + 4H^3)} ,$$

where

$D' = (a' - b')(b' - r')(a' - r')$, $H = -(3c + b^2)/9$ and $G = -(27d + 9bc + 2b^3)/27$.

Some identities will follow.

Identity 1. For q, q_1, q_2, q_3 and u denoting positive integers (where $q > 2$),

$$(2.5)\qquad \begin{vmatrix} T_q & T_{q-1} & T_{q-2} & T_{q+u} \\ T_{q_1} & T_{q_1-1} & T_{q_1-2} & T_{q_1+u} \\ T_{q_2} & T_{q_2-1} & T_{q_2-2} & T_{q_2+u} \\ T_{q_3} & T_{q_3-1} & T_{q_3-2} & T_{q_3+u} \end{vmatrix} = 0 .$$

Proof. Let

$$(2.6)\qquad T_{n+1}^{(1)} T_q + T_{u+1}^{(2)} T_{q-1} + T_u^{(3)} T_{q-2} - T_{q+u} = 0 .$$

Replace q by q_1, q_2 and q_3 in (2.6). Then we get

$$(2.7)\qquad \begin{aligned} &(a)\quad T_{u+1}^{(1)} T_{q_1} + T_{u+1}^{(2)} T_{q_1-1} + T_u^{(3)} T_{q_1-2} - T_{q_1+u} = 0 , \\ &(b)\quad T_{u+1}^{(1)} T_{q_2} + T_{u+1}^{(2)} T_{q_2-1} + T_u^{(3)} T_{q_2-2} - T_{q_2+u} = 0 , \\ &(c)\quad T_{u+1}^{(1)} T_{q_3} + T_{u+1}^{(2)} T_{q_3-1} + T_u^{(3)} T_{q_3-2} - T_{q_3+u} = 0 . \end{aligned}$$

Clearly, on eliminating $T_{u+1}^{(1)}$, $T_{u+1}^{(2)}$ and $T_u^{(3)}$ from (2.6), (2.7)a, (2.7)b and (2.7)c, the desired result follows. Note the following particular cases.

$$(2.8)\qquad \begin{vmatrix} T_q & T_{q-1} & T_{q-2} & T_{q+u} \\ T_{q+1} & T_q & T_{q-1} & T_{q+u+1} \\ T_{q+2} & T_{q+1} & T_q & T_{q+u+2} \\ T_{q+3} & T_{q+2} & T_{q+1} & T_{q+u+3} \end{vmatrix} = 0 ,$$

$$(2.9)\qquad \begin{vmatrix} T_q & T_{q-1} & T_{q-2} & T_{2q} \\ T_{q+1} & T_q & T_{q-1} & T_{2q+1} \\ T_{q+2} & T_{q-1} & T_q & T_{2q+2} \\ T_{q+3} & T_{q+2} & T_{q+1} & T_{2q+3} \end{vmatrix} = 0\,,$$

$$(2.10)\qquad \begin{vmatrix} T_q & T_{q-1} & T_{q-2} & T_{3q} \\ T_{q+1} & T_q & T_{q-1} & T_{3q+1} \\ T_{q+2} & T_{q+1} & T_q & T_{3q+2} \\ T_{q+3} & T_{q+2} & T_{q-1} & T_{3q+3} \end{vmatrix} = 0\,.$$

Identity 2. For q, q_1, q_2, $\cdots$, q_3 and u denoting positive integers (where $q + q_1 = r_1$, $q + q_2 = r_2$, $\cdots$, $q + q_9 = r_9$ and $q > 2$),

$$(2.11)\qquad \begin{vmatrix} T_q^2 & T_{q-1}^2 & T_{q-2}^2 & T_{q+u}^2 & T_qT_{q-1} & T_qT_{q-2} & \cdots & T_{q-2}T_{q+u} \\ T_{r_1}^2 & T_{r_1-1}^2 & T_{r_1-2}^2 & T_{r_1+u}^2 & T_{r_1}T_{r_1-1} & T_{r_1}T_{r_1-2} & \cdots & T_{r_1-2}T_{r_1+u} \\ \vdots & & & & & & & \\ T_{r_9}^2 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix} = 0.$$

The proof is left to the reader.

Identity 3. For q, q_1, q_2, q_3 and u denoting positive integers (where $q + q_1 = r_1$, $q + q_2 = r_2$, $q + q_3 = r_3$ and $q > 2$) if

$$A_{qq_1} = T_qT_{r_1} + T_{q-1}T_{r_1-1} + T_{q-2}T_{r_1-2} + T_{q+u}T_{r_1+u}\,,$$

then

$$(2.12)\qquad \begin{vmatrix} A_{q0} & A_{qq_1} & A_{qq_2} & A_{qq_3} \\ A_{qq_1} & A_{r_1+q_1} & A_{r_1+q_2} & A_{r_1,q_3} \\ A_{qq_2} & A_{r_2+q_1} & A_{r_2+q_2} & A_{r_2,q_3} \\ A_{qq_3} & A_{r_3+q_1} & A_{r_3+q_2} & A_{r_3,q_3} \end{vmatrix} = 0\,.$$

The proof is left to the reader.

Identity 4. For $q, q_1, q_2, \cdots, q_9$ and u denoting positive integers (where $q + q_1 = r_1,\ q + q_2 = r_2, \cdots, q + q_9 = r_9$ and $q > 2$) if

$$B_{qq_1} = T_q^2 T_{r_1}^2 + T_{q-1}^2 T_{r_1-1}^2 + \cdots + T_q T_{q-1} T_{r_1} T_{r_1-1} + \cdots + T_{q-2} T_{q+u} T_{r_1-2} T_{r_1+u}\,,$$

then

$$(2.13)\qquad \begin{vmatrix} B_{q_0} & B_{qq_1} & B_{qq_2} & B_{qq_3} & \cdots & B_{qq_9} \\ B_{qq_1} & B_{r_1,q_1} & B_{r_1,q_2} & B_{r_1,q_3} & \cdots & B_{r_1,q_9} \\ \vdots & & & & & \\ B_{qq_9} & B_{r_9,q_1} & B_{r_9,q_2} & B_{r_9,q_3} & \cdots & B_{r_9,q_9} \end{vmatrix} = 0\,.$$

The proof is left to the reader. We proceed to construct a field closely associated with $\{T_n\}_0^\infty$ which may be called hereafter the "Tribonacci field." The elements of this field are

$$(2.14)\qquad \frac{X^n}{Y^{n-2}}, \qquad n = 0, 1, 2, \cdots, \infty\,.$$

For $n \geq 3$, these elements modulo $X^3 - bX^2Y - cXY^2 - dY^3$ are the second-degree polynomials

$$(2.15)\qquad \frac{X^n}{Y^{n-2}} = T_{n-1}^{(1)} X^2 + T_{n-1}^{(2)} XY + dT_{n-2}^{(1)} Y^2\,.$$

In this field, if $P = Y^2$, $Q = XY$ and $R = X^2$, then the above-cited properties hold true.

3. PROPERTIES OF $\{T_n^{(4)}\}_0^\infty$

For this sequence,

(3.1) $$1 = (b - r)/D, \quad m = (r - a)/D, \quad n = (a - b)/D$$

and the n^{th} member is given by

(3.2) $$T_n^{(4)} = \frac{(b - r)a^n + (r - a)b^n + (a - b)r^n}{(a - b)(b - r)(a - r)}$$

or

(3.3) $$T_n^{(4)} = \frac{(b' - r')(a' + b/3)^n + (r' - a')(b' + b/3)^n + (a' - b')(r' + b/3)^n}{(a' - b')(b' - r')(a' - r')} \quad .$$

We simplify (3.2) and (3.3). Rewrite (3.2) as

$$T_n^{(4)} = \left[r^{n+1} - a^n r - br^n + ba^n - r^{n+1} + r^n a + rb^n - ab^n\right] \div \left[(a - b)(b - r)(a - r)\right]$$

(3.4) $$= \frac{1}{a - b}\left\{\frac{(r^n - a^n)(r - b) - (r - a)(r^n - b^n)}{(r - b)(r - s)}\right\}$$

$$= \frac{1}{a - b}\left\{\frac{r^n - a^n}{r - a} - \frac{r^n - b^n}{r - b}\right\} \quad .$$

This expression may be simplified as

(3.5) $$T_n^{(4)} = \frac{r^{n-1}}{a - b}\left\{\frac{1 - (ar^{-1})^n}{1 - (a/r)} - \frac{1 - (br^{-1})^n}{1 - (b/r)}\right\}$$

or

$$T_n^{(4)} = \frac{1}{a - b}\left\{r^{n-1}a + r^{n-2}a^2 + r^{n-3}a^3 + \cdots + ra^{n-1} - (r^{n-1}b + r^{n-2}b^2 + r^{n-3}b^3 + \cdots + rb^{n-1})\right\}$$

(3.6) $$= \frac{1}{a - b}\left\{r^{n-1}(a - b) + r^{n-2}(a^2 - b^2) + r^{n-3}(a^3 - b^3) + \cdots + r(a^{n-1} - b^{n-1})\right\}$$

$$= r^{n-1} + r^{n-2}(a + b) + r^{n-3}(a^2 + ab + b^2) + \cdots + r(a^{n-2} + a^{n-3}b + \cdots + b^{n-2}) .$$

Consider (3.3). Let

$$A_1 = \left\{\frac{-G + \sqrt{(G^2 + 4H^3)}}{2}\right\}^{1/3}, \qquad B_1 = \left\{\frac{-G - \sqrt{(G^2 + 4H^3)}}{2}\right\}^{1/3} \tag{3.7}$$

Clearly,

$$a' = A_1 + B_1, \qquad b' = wA_1 + w^2B_1 \qquad \text{and} \qquad b' = w^2A_1 + wB_1 \; . \tag{3.8}$$

On substituting for a', b' and r' from (3.8) in (3.3), we have

$$\begin{aligned} T_n^{(4)} &= [\{(w - w^2)A_1 + (w^2 - w)B_1\}(A_1 + B_1 + b/3)^n + \{(w^2 - 1)A_1 \\ &\quad + (w - 1)B_1\}(wA_1 + w^2B_1 + b/3)^n + \{(1 - w)A_1 + (1 - w^2)B_1\}(w^2A_1 \\ &\quad + wB_1 + b/3)^n] \div D \\ &= \Bigg[\sum_{r=0}^{r=n} {}_nC_r (b/3)^{n-r}\{(w - w^2)(A_1 + B_1)^r(A_1 - B_1) \\ &\quad + (wA_1 + w^2B_1)^r[(w^2A_1 + wB_1) - (A_1 + B_1)] + (w^2A_1 + wB_1)^r \\ &\quad \times [(A_1 + B_1) - (wA_1 + w^2B_1)]\}\Bigg] \div \{3(w - w^2)(A_1^3 - B_1^3)\} \\ &= \Bigg[\sum_{r=0}^{r=n} {}_nC_r (b/3)^{n-r}\{(w - w^2)(A_1 + B_1)^r(A_1 - B_1) + (A_1 + B_1) \\ &\quad \times [(w^2A_1 + wB_1)^r - (wA_1 + w^2B_1)^r] + (w^2A_1 + wB_1)(wA_1 + w^2B_1)^r \\ &\quad - (wA_1 + w^2B_1)(w^2A_1 + wB_1)^r\}\Bigg] \div \{3(w - w^2)(A_1^3 - B_1^3)\} \\ &= \Bigg[\sum_{r=0}^{r=n} {}_nC_r (b/3)^{n-r}\{(w - w^2)(A_1 + B_1)^r(A_1 - B_1) + (A_1 + B_1) \\ &\quad \times [(w^2A_1 + wB_1)^r - (wA_1 + w^2B_1)^r] - (A_1^2 - A_1B_1 + B_1^2)[(w^2A_1 \\ &\quad + wB_1)^{r-1} - (wA_1 + w^2B_1)^{r-1}]\}\Bigg] \div \{3(w - w^2)(A_1^3 - B_1^3)\} \; . \end{aligned} \tag{3.9}$$

Since

(3.10)
$$(wA_1 + w^2B_1)^r - (w^2A_1 + wB_1)^r = \frac{\sqrt{3}i}{2^r}\sum_{s=0}^{s=r} {}_rC_s i^{s-1} 3^{(s-1)/2} \times (A_1 + B_1)^{r-s}(A_1 - B_1)^s[1 - (-1)^s]$$

for $i = \sqrt{-1}$, $w = (1 + i\sqrt{3})/2$ and $w^2 = (1 - i\sqrt{3})/3$, (3.9) can be rewritten as

(3.11)
$$T_n^{(4)} = \left[\sum_{r=0}^{r=n} {}_nC_r (b/3)^{n-r}\left\{ i\sqrt{3}(A_1 + B_1)^r(A_1 - B_1) - (A_1 + B_1)\right.\right.$$
$$\times\left[\frac{\sqrt{3i}}{2^r}\sum_{s=0}^{s=r} {}_rC_s i^{s-1}3^{(s-1)/2}(A_1 + B_1)^{r-s}(A_1 - B_1)^s\right.$$
$$\left.\times\ (1 - (-1)^s)\right] + (A_1^2 - A_1B_1 + B_1^2)\left[\frac{\sqrt{3i}}{2^{r-1}}\sum_{s=0}^{s=r-1} {}_{r-1}C_s i^{s-1}\right.$$
$$\left.\left.\left.\times\ 3^{(s-1)/2}(A_1 + B_1)^{r-s-1}(A_1 - B_1)^s(1 - (-1)^s)\right]\right\}\right]$$
$$\div\ [3i\sqrt{3}(A_1^3 - B_1^3)]$$
$$= \left[\sum_{r=0}^{r=s} {}_nC_r (b/3)^{n-r}\left\{(A_1 + B_1)^r(A_1 - B_1) - (A_1 + B_1)\right.\right.$$
$$\times\left[\frac{1}{2^r}\sum_{r=0}^{r=s} {}_rC_s i^{s-1}3^{(s-1)/2}(A_1 + B_1)^{r-s}(A_1 - B_1)s\right.$$
$$\left.\times\ (1 - (-1)^s)\right] + (A_1^2 - A_1B_1 + B_1^2)\left[\frac{1}{2^{r-1}}\sum_{s=0}^{s=r-1} {}_{r-1}C_s i^{s-1}\right.$$
$$\left.\left.\left.\times\ 3^{(s-1)/2}(A_1 + B_1)^{r-s-1}(A_1 - B_1)^s(1 - (-1)^s)\right]\right\}\right]$$
$$\div\ \{3(A_1^3 - B_1^3)\}\ .$$

However on combining (2.3) and (3.2),

(3.12)
$$
\begin{aligned}
T_n^{(4)} &= [\{(w - w^2)A^{1/3} + (w^2 - w)B^{1/3}\}(A^{1/3} + B^{1/3} + b/3)^n \\
&\quad + \{(w^2 - 1)A^{1/3} + (w - 1)B^{1/3}\}(wA^{1/3} + w^2B^{1/3} + b/3)^n \\
&\quad + \{(1 - w)A^{1/3} + (1 - w^2)B^{1/3}\}(w^2A^{1/3} + wB^{1/3} + b/3)^n] \\
&\quad \div [3(w - w^2)(A - B)] \\
&= \sum_{\delta=0}^{\delta=n} (b/3^{n-\delta}\,{}_nC_\delta L_\delta ,
\end{aligned}
$$

where

$$
L_{3k} = \left[\sum_{r=o}^{r=k-1} A^{(3k-3r-1)/3} B^{(3r+2)/3} ({}_{3k}C_{3r+1} - {}_{3k}C_{3r+2})\right] \div (B - A),
$$

$$
L_{3k+1} = \left[\sum_{r=0}^{r=k} A^{(3k+1-3r)/3} B^{(3r+1)/3} ({}_{3k+1}C_{3r} - {}_{3k+1}C_{3r+1})\right] \div (B - A)
$$

and

$$
\begin{aligned}
L_{3k+2} = \Bigg[& B^{(3k+3)/2} - A^{(3k+3)/3} + \sum_{r=0}^{r=k-1} A^{(3k-3r)/3} \\
& B^{(3r+3)/3} ({}_{3k+2}C_{3r+2} - {}_{3k+2}C_{3r+3}) \Bigg] \div (B - A) .
\end{aligned}
$$

4. PROPERTIES OF $\{T_n^{(5)}\}_0^\infty$

Let

(4.1)
$$
\begin{aligned}
T_n^{(5)} &= \frac{(b - a)(b - r)a^n + (b - b)(r - a)b^n + (b - r)(a - b)r^n}{D} \\
&= [b\{(b - r)a^n + (r - a)b^n + (a - b)r^n\} - \{(b - r)a^{n+1} \\
&\quad + (r - a)b^{n+1} + (a - b)r^{n+1}\}] \div D \\
&= bT_n^{(4)} - T_{n+1}^{(4)} .
\end{aligned}
$$

This relation is useful in deriving expressions of $T_n^{(5)}$ similar to those of $T_n^{(4)}$.

5. PROPERTIES OF $\{T_n^{(6)}\}_0^\infty$

Proceeding as in previous section, it is easy to show that

$$T_n^{(6)} = dT_{n-1}^{(4)} \quad . \tag{5.1}$$

Note that this equation connects up expressions of $T_n^{(4)}$ in Section 3.

6. PROPERTIES OF $\{T_n^{(7)}\}_0^\infty$

In this Section, we state without proof the following identities:

$$2(T_{3n}^{(7)} - 3d^n) = T_n^{(7)}\{2T_{2n}^{(7)} - (T_n^{(7)})^2 + T_{2n}^{(7)}\}, \tag{6.1}$$

$$T_{4n}^{(7)} = T_n^{(7)}T_{3n}^{(7)} - T_{2n}^{(7)}[\{(T_n^{(7)})^2 - T_{2n}^{(7)}\}/2] + d^nT_n^{(7)}, \tag{6.2}$$

$$\begin{aligned} T_{4n+4r}^{(7)} - T_{4n}^{(7)} = {} & T_{n+r}^{(7)}T_{3n+3r}^{(7)} - T_n^{(7)}T_{3n}^{(7)} - T_{2n+2r}^{(7)}[\{(T_{n+r}^{(7)})^2 \\ & - 2T_{2n+2r}^{(7)}\}/2] + T_{2n}^{(7)}[\{(T_n^{(7)})^2 - 2T_{2n}^{(7)}\}/2] \\ & + d^n(T_{n+r}^{(7)} - T_n^{(7)}), \end{aligned} \tag{6.3}$$

$$2T_{3n}^{(7)} = 3T_n^{(7)}T_{2n}^{(7)} + 6d^n - (T_n^{(7)})^3 = T_n^{(7)}\{3T_{2n}^{(7)} - (T_n^{(7)})^2\} + 6d^n \tag{6.4}$$

and

$$(T_{n+r}^{(7)})^3 - (T_n^{(7)})^3 = 3\{T_{n+r}^{(7)}T_{2n+2r}^{(7)} - T_n^{(7)}T_{2n}^{(7)}\} - 2\{T_{3n+3r}^{(7)} - T_{3n}^{(7)}\} \tag{6.5}$$

for $d = 1$.

7. PROPERTIES OF $\{T_n^{(8)}\}_0^\infty$

This section will give identities relating to $\{T_n^{(8)}\}_0^\infty$ and $\{T_n^{(9)}\}_0^\infty$. They are:

$$T_n^{(9)} - T_n^{(8)} = T_{n-3}^{(9)} , \tag{7.1}$$

(7.2) $$T_n^{(9)} + T_n^{(8)} = T_{n+1}^{(9)} ,$$

(7.3) $$(T_n^{(9)})^2 - (T_n^{(8)})^2 = T_{n-3}^{(9)} T_{n+1}^{(9)} ,$$

(7.4) $$2(T_n^{(9)})^2 = T_n^{(9)}\{T_{n-3}^{(9)} + T_{n+1}^{(9)}\}$$

and

(7.5) $$4T_n^{(9)}T_n^{(8)} = (T_{n+1}^{(9)})^2 - (T_{n-3}^{(9)})^2 .$$

8. PROPERTIES OF $\{T_n^{(9)}\}_0^\infty$

This section will discuss the congruence properties of $\{T_n^{(9)}\}_0^\infty$ modulo m, a positive integer. We note the following identity:

(8.1) $$T_{q+u}^{(9)} = T_{u+2}^{(9)}T_q^{(9)} + (T_{u+1}^{(9)} + T_u^{(9)})T_{q-1}^{(9)} + T_{u+1}^{(9)}T_{q-2}^{(9)} .$$

Some theorems concerning $\{T_n^{(9)} \pmod m\}_0^\infty$ will follow.

Theorem a. $\{T_n^{(9)} \pmod m\}_0^\infty$ is simply periodic.

Proof. For some n and a, let $T_{n-1}^{(9)} \equiv T_{n-1}^{(9)} \pmod m$, $T_n^{(9)} \equiv T_a^{(9)} \pmod m$ and $T_{n+1}^{(9)} \equiv T_{a+1}^{(9)} \pmod m$. From these congruences, we obtain

(8.2) $$T_{n\pm t}^{(9)} \equiv T_{a\pm t}^{(9)} \pmod m ,$$

where t denotes an integer ≥ 2. Since m^2 pairs of terms are possible in this series, $\{T_n^{(9)} \pmod m\}_0^\infty$ must return to the starting values thus becoming simply periodic.

Theorem b. For a prime factorization of m in the form $m = \Pi p_i^{e_i}$, the period of $\{T_n^{(9)} \pmod m\}_0^\infty$ is the lowest common multiple of all the periods of

$$\{T_n^{(9)} \pmod{p_i^{e_i}}\}_0^\infty .$$

Proof. Let k(m) denote the period of $\{T_n^{(9)} \pmod m\}_0^\infty$. Then k(m) is of the form

$$c_i k(p_i^{e_i}) ,$$

where c_i denotes a related constant. Therefore,

$$k(m) = \text{l.c.m.}\left[k(p_i^{e_i}) \right] .$$

Theorem c. For some q, if $T_q^{(9)} \equiv 0 \pmod m$ and $T_{q+1}^{(9)} \equiv 0 \pmod m$, then the subscripts for which $T_n^{(9)} \equiv 0 \pmod m$ and $T_{n+1}^{(9)} \equiv 0 \pmod m$ form simple arithmetic progressions.

Proof. Let

$$\begin{aligned} T_{q+q'+1}^{(9)} &\equiv T_{q'+2}^{(9)} T_{q+1}^{(9)} + (T_{q'+1}^{(9)} + T_{q'}^{(9)}) T_q^{(9)} + T_{q'+1}^{(9)} T_{q-1}^{(9)} \\ &\equiv T_{q'+1}^{(9)} T_{q-1}^{(9)} \pmod m . \end{aligned} \tag{8.3}$$

For $q' = q - 1$ and q, this congruence shows that $T_{2q}^{(9)} \equiv 0 \pmod m$ and $T_{2q+1}^{(9)} \equiv 0 \pmod m$. Similarly, we can obtain

$$T_{3q}^{(9)} \equiv 0 \pmod m, \qquad T_{3q+1}^{(9)} \equiv 0 \pmod m ,$$

$$T_{4q}^{(9)} \equiv 0 \pmod m, \qquad T_{4q+1}^{(9)} \equiv 0 \pmod m, \quad \text{etc.}$$

Therefore, it follows that n is of the form xq, $x = 1, 2, \cdots$, so that n and n + 1 form simple arithmetic progressions.

Theorem d. Let $H' = 3^2 H$, $G' = 3^3 G$ and $(G')^2 + 4(H')^3$ be a quadratic residue for primes of the form $3t - 1$. Then $k(p) \mid (p^2 - 1)$.

Proof. Denote p^2 by $3t' + 1$. Then

$$L_{3t'+1} \equiv 3^{3t'}L_{3t'+1} \equiv 3^{3t'}\left[\left\{\sum_{r=0}^{r-t'} A^{(3t'+1-3r)/3}\right.\right.$$

(8.4)
$$\left.\left. \times B^{(3r+1)/3}({}_{3t'+1}C_{3r-3t'+1}\ C_{3r+1})\right\} \div (B - A)\right] .$$

$$\equiv 3^{3t'}(A^{(3t'+1)/3}B^{1/3} - B^{(3t'+1)/3}A^{1/3}) \div (B - A)$$

$$\equiv 3H'U'_{t'} \pmod p \quad ,$$

$$L_{3t'+2} \equiv 3^{3t'}L_{3t'+2} \equiv 3^{3t'}\left[\left\{B^{(3t'+3)/3} - A^{(3t'+3)/3}\right.\right.$$

(8.5)
$$\left.\left. + \sum_{r=0}^{r=t'-1} A^{t--r}B^{r+1}({}_{3t'+2}C_{3r+2-3t'+2}C_{3r+3})\right\} \div (B - A)\right]$$

$$\equiv 3^{3t'}(B^{t'+1} - A^{t'+1}) \div (B - A)$$

$$\equiv U'_{t'+1} \pmod p$$

and

$$L_{3t'+3} \equiv 3^{3t'}\left\{\sum_{r=0}^{r=t'} A^{(3t'+2-3r)/3}B^{(3r+2)/3}({}_{3t'+3}C_{3r+1-3t'+3}\ C_{3r+2})\right\}$$

(8.6)
$$\div (B - A)$$

$$\equiv 3^{3t'}(A^{2/3}B^{(3t'+2)/3} - B^{2/3}A^{(3t'+2)/3}) \div (B - A)$$

$$\equiv (1/3)(H')^2U'_{t'} \pmod p \quad ,$$

so that

$$T^{(9)}_{3t'+1} \equiv 3H'U'_t \pmod p$$

(8.7)
$$T^{(9)}_{3t'+2} \equiv H'U'_{t'} + U'_{t'+1} \pmod p$$

and

$$T^{(9)}_{3t'+3} \equiv (1/3) + (2/3)U'_{t'+1} + (1/3)(H')^2U'_{t'} + (1/3)H'U'_{t'} \pmod p ,$$

where
$$U'_{n+1} \equiv -G'U'_n + (H')^3 U'_{n-1}$$

for $n = 1, 2, \cdots$, $U'_0 = 0$ and $U'_1 = 1$. Since

$$U'_{3t-2} \equiv 0 \pmod p \quad \text{and} \quad U'_{(3t-2)+1} \equiv 1 \pmod p,$$

we get

$$U'_{t'} \equiv 0 \pmod p \quad \text{and} \quad U'_{t'+1} \equiv 1 \pmod p .$$

Therefore

$$(8.8) \quad T^{(9)}_{3t'+1} \equiv 0 \pmod p, \quad T^{(9)}_{3t'+2} \equiv 1 \pmod p \quad \text{and} \quad T^{(9)}_{3t'+3} \equiv 1 \pmod p.$$

These congruences imply

$$T^{(9)}_{3t'} \equiv 0 \pmod p \quad \text{and} \quad k(p) \mid (p^2 - 1) .$$

Theorem e. For primes of the form $p = 3t - 1$ where $(G')^2 + 4(H')^3$ is a quadratic nonresidue, $k(p) \mid (p^6 - 1)$.

Let $p^6 = 3t'' + 1$. Note that the proof of Theorem 4 holds with t' changed to t'', etc. The proof is left to the reader.

Theorem f. For primes of the form $p = 3t + 1$ where $(G')^2 + 4(H')^3$ is a quadratic nonresidue, $k(p) \mid (p^2 - 1)$.

Proof. Let $p^2 = 3t + 1$. Then

$$T^{(9)}_{3t+1} \equiv 3^{3t} T^{(9)}_{3t+1} \equiv 3^{3t} \sum_{\delta=0}^{\delta=3t+1} (1/3)^{3t+1-\delta} {}_{3t+1}C_\delta L_\delta$$

$$\equiv 3^{3t} L_{3t+1}$$

$$(8.9) \quad \equiv 3^{3t} \left[\left\{ \sum_{r=0}^{r=t} A^{(3t+1-3r)/3} B^{(3r+1)/3} ({}_{3t+1}C_{3r} - {}_{3t+1}C_{3r+1}) \right\} \div (B - A) \right]$$

$$\equiv 3^{3t} (A^{(3t+1)/3} B^{1/3} - A^{1/3} B^{(3t+1)/3})/(B - A)$$

$$\equiv 3^{3t} A^{1/3} B^{1/3} (A^t - B^t)/(B - A)$$

$$\equiv 3H'U'_t \pmod p ,$$

$$T^{(9)}_{3t+2} \equiv 3^{3t}T^{(9)}_{3t+2} \equiv 3^{3t}\sum_{\delta=0}^{\delta=3t+2}(1/3)^{3t+2-\delta}\,{}_{3t+2}C_\delta L_\delta$$

$$\equiv 3^{3t}\{(1/3)L_{3t+1} + L_{3t+2}\}$$

(8.10)
$$\equiv H'U'_t + 3^{3t}\left[B^{t+1} - A^{t+1} + \sum_{r=0}^{r=t-1} A^{(3t-3r)/3}B^{(3r+3)/2}\right.$$
$$\left.({}_{3t+2}C_{3r+2} - {}_{3t+2}C_{3r+3}) \div (B - A)\right]$$
$$\equiv H'U'_t + 3^{3t}(B^{t+1} - A^{t+1})/(B - A)$$
$$\equiv H'U'_t + U'_{t+1} \pmod p$$

and

$$T^{(9)}_{3t+3} \equiv 3^{3t}T^{(9)}_{3t+3} \equiv 3^{3t}\sum_{\delta=0}^{\delta=3t+3}(1/3)^{3t+3-\delta}\,{}_{3t+3}C_\delta L_\delta$$

$$\equiv 3^{3t}\{(1/3)^{3t+3}L_0 + (1/3)^{3t+2}L_1 + (1/3)^{3t+1}L_2$$
$$+ L_2(1/3)^2L_{3t+1} + (2/3)L_{3t+2} + L_{3t+3}\}$$

(8.11)
$$\equiv (1/3) + (1/3)H'U'_t + (2/3)U'_{t+1} + 3^{3t}\left[\sum_{r=0}^{r=t} A^{(3t+2-3r)/3}\right.$$
$$\left.B^{(3r+2)/3}({}_{3t+3}C_{3r+1} - {}_{3t+3}C_{3r+2})\right] \div (B - A)$$
$$\equiv (1/3) + (1/3)H'U'_t + (2/3)U'_{t+1} + 3^{3t}$$
$$\{A^{2/3}B^{2/3}(B^t - A^t)/(B - A)\}$$
$$\equiv (1/3) + (1/3)H'U'_t + (2/3)U'_{t+1} + (1/3)(H')^2U'_t \pmod p \quad .$$

For the considered primes, it is easy to show that

(8.12)
$$U'_{3t+2} \equiv 0 \pmod p, \quad U'_{(3t+2)+1} \equiv \{(-H')^3\} \pmod p ,$$
$$U'_{2(3t+2)} \equiv 0 \pmod p, \quad U'_{2(3t+2)+1} \equiv \{(-H')^3\}^2 \pmod p ,$$
$$\cdots \qquad \cdots \qquad \cdots \qquad \cdots$$
$$U'_{t(3t+2)} \equiv 0 \pmod p, \quad U'_{t(3t+2)+1} \equiv \{(-H')^3\}^t \pmod p ,$$

so that

$$U'_t \equiv 0 \pmod{p} \qquad \text{and} \qquad U'_{t+1} \equiv 2^{3t} \equiv 1 \pmod{p}.$$

Therefore

$$(8.13) \quad T^{(9)}_{3t+1} \equiv 0 \pmod{p}, \quad T^{(9)}_{3t+2} \equiv 1 \pmod{p} \quad \text{and} \quad T^{(9)}_{3t+3} \equiv 1 \pmod{p},$$

when $T^{(9)}_{3t} \equiv 0 \pmod{p}$ and the desired result follows.

Theorem g. For primes of the form $p = 3t + 1$ where $(G')^2 + 4(H')^3$ is a quadratic residue, $k(p) \mid (p^6 - 1)$.

Let $p^6 = 3t' + 1$. Note that the proof of Theorem f holds with t changed to t', etc. The proof is left to the reader.

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