ALGORITHM FOR ANALYZING A LINEAR RECURSION SEQUENCE

BROTHER ALFRED BROUSSEAU
St. Mary's College, California

The basic idea of the algorithm to be presented in this paper may be illustrated by the simple example of a third-order linear recursion sequence: 1, 4, 8, 21, 67, 199, 568, 1641, 4782, 13904, 40353, 117161, ··· with a recursion relation of the form

$$T_{n+1} = aT_n + bT_{n-1} + cT_{n-2} .$$

The problem is to find a, b, c. The obvious procedure is to set up a set of linear equations

$$L_1: \quad c + 4b + 8a = 21$$
$$L_2: \quad 4c + 8b + 21a = 67$$
$$L_3: \quad 8c + 21b + 67a = 199$$
$$L_4: \quad 21c + 67b + 199a = 568 .$$

Only three equations are needed to find a, b, and c; the fourth is introduced since it brings out the fact that it must be a linear combination of the first three, the multipliers being in fact the quantities a, b, c for the given sequence; that is,

$$L_4 = aL_3 + bL_2 + cL_1$$

Hence we can ascertain that the sequence is of the third order by fact that these four relations are linearly dependent and no smaller number has this property. Thereafter, the first three equations can be used to find the quantities a, b, c.

The algorithm that does both these jobs simultaneously is Gaussian elimination. Stripped of the excess baggage we start with a matrix of quantities:

(1a)	1	4	8	21
(2a)	4	8	21	67
(3a)	8	21	67	199
(4a)	21	67	199	568

Multiply the first set of quantities by -4 and the second by 1 and add to get (2b); multiply the first set by -8 and the third by 1 to get (3b); multiply the first by -21 and the fourth by 1 to get (4b).

(2b)	0	-8	-11	-17
(3b)	0	-11	3	31
(4b)	0	-17	31	127

Now use -8 in (2b) as the pivot value and eliminate -11 and -17 in (3b) and (4b).

(3c)	0	0	-145	-435
(4c)	0	0	-435	-1305

Finally by another elimination

(4d)	0	0	0	0

This shows that there is a third-order linear recursion relation among the quantities we have used in these equations.

To find the constants a, b, c, we have equivalently from (4c):

$$-435a = -1305 \quad \text{so that} \quad a = 3 .$$

Then from (3b) $-11b + 3c = 31$ which gives $b = -2$. Finally from (1a), $c + 4b + 8a = 21$ we have $c = 5$. The recursion relation in question is:

$$T_{n+1} = 3T_n - 2T_{n-1} + 5T_{n-2} .$$

It can now be ascertained whether the remaining terms are governed by this recursion relation.

In general, given a linear recursion relation for which we do not know the order or the coefficients, we can proceed by Gaussian elimination until we find a row of zeros. If this is the n^{th} row, then the order of the linear recursion relation is $n - 1$. The coefficients can then be found by back-substitution as was done in the illustrative example.

For example, the sequence $\cdots 527, 110, 23, 5, 2, 5, 23, 110, 527, \cdots$ was obtained as a fourth-order sequence. Is it a proper fourth-order sequence or does it have a lower-order factor which governs the sequence? We proceed to make our analysis.

(1a)	527	110	23	5	2
(2a)	110	23	5	2	5
(3a)	23	5	2	5	23
(4a)	5	2	5	23	110
(5a)	2	5	23	110	527

(2b)	0	21	105	504	2415
(3b)	0	105	525	2520	12075
(4b)	0	504	2520	12096	57960
(5b)	0	2415	12075	57960	277725
(3c)	0	0	0	0	0

The sequence is governed by a second-order recursion relation:

$$T_{n+1} = 5T_n - T_{n-1} .$$

As a more ambitious example, consider the sequence:

77, -20, 1, 0, -8, -2, 5, -2, 1, 9, 1, -2, 5, -2, -8, 0, 1, -20, 77, -425, ··· ,

which is supposed to be of the sixth order.

(1a)	1	0	-8	-2	5	-2	1	
(2a)	0	-8	-2	5	-2	1	9	
(3a)	-8	-2	5	-2	1	9	1	
(4a)	-2	5	-2	1	9	1	-2	
(5a)	5	-2	1	9	1	-2	5	
(6a)	-2	1	9	1	-2	5	-2	
(7a)	1	9	1	-2	5	-2	-8	
(2b)	0	-8	-2	5	-2	1	9	
(3b)	0	-2	-59	-18	41	-7	9	
(4b)	0	5	-18	-3	19	-3	0	
(5b)	0	-2	41	19	-24	8	0	
(6b)	0	1	-7	-3	8	1	0	
(7b)	0	9	9	0	0	0	-9	
(3c)	0	0	468	154	-332	58	-54	
(4c)	0	0	154	-1	-142	19	-45	
(5c)	0	0	-332	-142	188	-62	18	
(6c)	0	0	58	19	-62	-9	-9	
(7c)	0	0	-54	-45	18	-9	-9	
(4d)	0	0	0	-3023	-1916	-5	-1593	
(5d)	0	0	0	-1916	-2780	-1220	-1188	
(6d)	0	0	0	-5	-1220	-947	-135	
(7d)	0	0	0	-1593	-1188	-135	-891	

(5e)	0	0	0	0	4732884	3678480	539136
(6e)	0	0	0	0	3678480	2862756	400140
(7e)	0	0	0	0	539136	400140	155844

(6f)	0	0	0	0	0	17876957904	-89384789520
(7f)	0	0	0	0	0	-89384789520	446923947600

(7g)	0	0	0	0	0	0	0

Back substitution gives for the coefficients -5, 4, -2, 4, -5, 1, so that the recursion relation is $T_{n+1} = -5T_n + 4T_{n-1} - 2T_{n-2} + 4T_{n-3} - 5T_{n-4} - T_{n-5}$.

A complication can arise when there is a zero in a position at which the pivot should be found. The general procedure here is to take as pivot in a given column the first candidate among the sets of coefficients that might serve as a possible pivot in this column.

The following example will illustrate the manner of proceeding. Let there be a sequence of unknown order: 1, 2, 4, 8, 11, 7, -11, -47, -94, -123, -76, 123, $\cdots$. We set up quantities to cover up to the fifth order as an initial guess.

(1a)	1	2	4	8	11	7	-11
(2a)	2	4	8	11	7	-11	-47
(3a)	4	8	11	7	-11	-47	-94
(4a)	8	11	7	-11	-47	-94	-123
(5a)	11	7	-11	-47	-94	-123	-76
(6a)	7	-11	-47	-94	-123	-76	123

(2b)	0	0	0	-5	-15	-25	-25
(3b)	0	0	-5	-25	-55	-75	-50
(4b)	0	-5	-25	-75	-135	-150	-35
(5b)	0	-15	-55	-135	-215	-200	45
(6b)	0	-25	-75	-150	-200	-125	200

The first pivot element in the second column occurs in (4b). So we carry down (2b) and (3b) and pivot on (4b).

(2c)	0	0	0	-5	-15	-25	-25
(3c)	0	0	-5	-25	-55	-75	-50
(5c)	0	0	-100	-450	-950	-1250	-750
(6c)	0	0	-250	-1125	-2375	-3125	-1875

The first pivot element in the third column is found in (3c).

[Continued on page 438.]