# PELL NUMBER TRIPLES

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Horadam [1] has shown that Pythagorean triples are Fibonacci-number triples. It has now been found that Pythagorean triples are Pell-number triples as well.

The Diophantine solution for Pythagorean triples $(x, y, z)$ is $x = 2pq$, $y = p^2 - q^2$, and $z = p^2 + q^2$, where $p > q$. For primitive solutions $(p,q) = 1$, $p$ and $q$ are of different parity, $x$ or $y \equiv 0 \pmod 3$ [2], $x \equiv 0 \pmod 4$, and all prime factors of $z$ are congruent to 1 modulo 4. Since $x \neq y$, regardless of primitivity, let

$$(1) \qquad y - x = p^2 - q^2 - 2pq = \pm c$$

which is readily transformed into

$$(2) \qquad p - q = \sqrt{2q^2 \pm c}$$

and

$$(3) \qquad p + q = \sqrt{2p^2 \pm c} \ .$$

It may be noted in passing that all values of $c$ for primitive triples are of the form $12d \pm 1$ and $12d \pm 5$, where $d = 0, 1, 2, 3, \cdots$. However, less than fifty percent of numbers of this form are possible values of $c$, because this representation by means of three Pell numbers includes all odd numbers not divisible by 3.

Two characteristic identities of the Pell-number sequence,

$$(4) \qquad P_{n+2} = 2P_{n+1} + P_n \qquad (P_0 = 0,\ P_1 = 1)$$

and

$$(5) \qquad (P_{n+1} + P_n)^2 - 2P_{n+1}^2 = (-1)^{n+1}$$

were used [3] to prove that Pell numbers generate all values for $(x, y, z)$ when $c = 1$. Multiplication of (5) by $a^2$ shows that Pell numbers also generate all values for $(x, y, z)$ when $c = a^2$, regardless of primitivity. Thus, when $c = 1$, $q_n = P_n$ and $p_n = P_{n+1}$; $c = 4$, $q_n = 2P_n$; $c = 9$, $q_n = 3P_n$, etc. Similarly, Pell numbers generate all $(x, y, z)$ when $c = 2a^2$, obviously nonprimitive. When $c = 2$, $q_n = P_{n+1} + P_n$ and $p_n = P_{n+2} + P_{n+1}$; $c = 8$, $q_n = 2(P_{n+1} + P_n)$; $c = 18$, $q_n = 3(P_{n+1} + P_n)$, etc.

All other Pythagorean triples are represented by generalized Pell numbers, similar to Horadam's generalized Fibonacci numbers [4], in such a way that a pair of equations is associated with each value of $c$.

(6a) $$q_{2n+1} \text{ or } q_{2n+2} = aP_{n+1} - bP_n$$

and

(6b) $$q_{2n+2} \text{ or } q_{2n+1} = bP_{n+1} + aP_n$$

where $a > b$. The value of $p$, associated with a given value of $q$, is obtained by replacing $n$ by $(n + 1)$. It will be noted that the odd and even values form two distinct sequences.

Upon combining (2) and (3) with (6), we obtain

(7a) $$p_{2n+1} - q_{2n+1} = (a - b)P_{n+1} + (a + b)P_n$$

(7b) $$p_{2n+2} - q_{2n+2} = (a + b)P_{n+1} + (b - a)P_n$$

and

(8a) $$p_{2n+1} + q_{2n+1} = (3a - b)P_{n+1} + (a - b)P_n$$

(8b) $$p_{2n+2} + q_{2n+2} = (a + 3b)P_{n+1} + (a + b)P_n ,$$

where the subscripts for $p$ and $q$ may be interchanged between (7a) and (7b) as well as between (8a) and (8b) as needed.

Since Pell numbers proper, and generalized Pell numbers for primitive solutions, are alternately of different parity, and with $p \pm q$ odd for primitive solutions, $a \pm b$ must be odd in view of (7) and (8). All other possible values of $a \pm b$ also occur and give rise to non-primitive triples. Thus, all $(x, y, z)$ can be generated, and no impossible values occur. Once obtained, all values are easily verified, and any oversight of a permissible value of $c$ becomes obvious by the absence of an expected pair $(a,b)$. But there appears to be no systematic, analytical method of determining a priori either possible values of $c$ or their associated pair or pairs of $(a,b)$, except for $c = a^2$ and $c = 2a^2$, where $b = 0$.

Following is a table of the first 33 values for $c$, $a$, and $b > 0$. Values of $c$ giving rise to primitive solutions are underlined.

| c | a | b | c | a | b | c | a | b |
|---|---|---|---|---|---|---|---|---|
| 7 | 2 | 1 | 56 | 6 | 2 | 97 | 7 | 6 |
| 14 | 3 | 1 | 62 | 7 | 1 | 98** | 9 | 1 |
| 17 | 3 | 2 | 63 | 6 | 3 | 103 | 8 | 3 |
| 23 | 4 | 1 | 68 | 6 | 4 | 112 | 8 | 4 |
| 28 | 4 | 2 | 71 | 6 | 5 | 113 | 9 | 2 |
| 31 | 4 | 3 | 73 | 7 | 2 | 119*** | 8 | 5 |
| 34 | 5 | 1 | 79 | 8 | 1 | 119*** | 10 | 1 |
| 41 | 5 | 2 | 82 | 7 | 3 | 124 | 8 | 6 |
| 46 | 5 | 3 | 89 | 7 | 4 | 126 | 9 | 3 |
| 47 | 6 | 1 | 92 | 8 | 2 | 127 | 8 | 7 |
| 49* | 5 | 4 | 94 | 7 | 5 | 136 | 10 | 2 |

*This also has the solution $7P_n$.
**This also has the solution $7(P_{n+1} + P_n)$

*** This is the first value with two pairs of solutions.

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