PERFECT N-SEQUENCES FOR N, N + 1, AND N + 2

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Frank S. Gillespie and W. R. Utz [1] define a (generalized) perfect n-sequence for m (where \( n \geq 2, m \geq 2 \)) to be a sequence of length \( mn \) in which each of the integers 1, 2, 3, \( \cdots, m \) occurs exactly \( n \) times and between any two occurrences of the integer \( x \) there are \( x \) entries. Examples of perfect 2-sequences are numerous: 3 1 2 1 3 2 for \( m = 3 \) and 4 1 3 1 2 4 3 2 for \( m = 4 \) are the simplest. However, the author knows of no perfect n-sequence if \( n > 2 \).

No perfect n-sequence for \( m \) exists if \( m \leq n \) [1]. (This is a direct corollary of Lemma 1, below.) It will be proved here that no perfect n-sequence for \( m \) exists if \( m = n \), \( m = n + 1 \), or \( m = n + 2 \) (except for the perfect 2-sequences for 3 and 4), extending the result slightly.

In a perfect n-sequence for \( m \), if \( x \) is an integer and \( 1 \leq x \leq m \), then there are \( n \) \( x \)'s in the sequence. The positions in the sequence will be numbered, in order, starting at the left, 1, 2, 3, \( \cdots, m \). Let \( p(x, i) \) mean "the position of the \( i \)th occurrence of the integer \( x \)". The first occurrence of an integer will have special significance; let \( P_x = p(x, 1) \).

Example. In the sequence 1 7 1 2 6 4 2 5 3 7 4 6 3 5, \( p(6, 1) = P_6 = 5 \), \( p(4, 2) = 11 \), \( P_2 = 4 \), etc.

Note that \( p(x, i) \) is meaningful if \( 1 \leq x \leq m \) and \( 1 \leq i \leq n \), and \( P_x \) is meaningful if \( 1 \leq x \leq m \).

In a perfect n-sequence for \( m \)

\[
\begin{align*}
(1) \quad p(x, i) &= P_x + (x + 1)(i - 1) \\
& \quad (1 \leq x \leq m; \ 1 \leq i \leq n)
\end{align*}
\]

which follows from the recursive formula (for \( i \geq 2 \))

\[
(2) \quad p(x, i) = p(x, i - 1) + (x + 1).
\]

Theorem 1. There is no perfect n-sequence for \( n \).

Proof. Assume such a sequence exists. Then it has \( n^2 \) entries. Also

\[
p(n, n) = P_n + (n + 1)(n - 1) = P_n + n^2 - 1
\]

so that \( P_n \) must be 1.

It is impossible that \( 1 \leq P_{n-1} \leq n \); otherwise \( p(n - 1, P_{n-1}) \) and \( p(n, P_{n-1}) \) are meaningful and
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\[ p(n - 1, P_{n-1}) = P_{n-1} + nP_{n-1} - n = p(n, P_{n-1}) \]

using (1) and \( P_n = 1 \). But this is impossible since an \( n \) and an \( n - 1 \) cannot occupy the same position.

It is impossible that \( n + 1 \leq P_{n-1} \); otherwise \( p(n - 1, n) \geq n^2 + 1 \), but the largest position is \( n^2 \).

Now \( 1 \leq n - 1 \leq n \) (since \( n \geq 2 \)) so that \( P_{n-1} \) is a positive integer, and we have a contradiction.

**Theorem 2.** There is no perfect \( n \)-sequence for \( n + 1 \), except the perfect 2-sequence for 3.

**Proof.** Assume such a sequence exists. Then there are \( n(n + 1) = n^2 + n \) entries.

Also,

\[ p(n + 1, n) = P_{n+1} + n^2 + n - 2 \]

so that either \( P_{n+1} = 1 \) or \( P_{n+1} = 2 \). If \( P_{n+1} = 2 \), then \( p(n + 1, n) = n^2 + n \), the last position, but since a perfect sequence taken in reverse order is still a perfect sequence, this case is symmetrical to the case \( P_{n+1} = 1 \). Hence only the case \( P_{n+1} = 1 \) need be considered.

It is impossible that \( 1 \leq P_n \leq n \); otherwise \( p(n, P_n) = p(n + 1, P_n) \). It is impossible that \( n + 2 \leq P_n \); otherwise \( p(n, n) \geq n^2 + n + 1 \). Therefore the only possibility is \( P_n = n + 1 \). Now we have \( P_{n+1} = 1 \) and \( P_n = n + 1 \).

It is impossible that \( 1 \leq P_{n-1} \leq n - 1 \); otherwise \( p(n - 1, P_{n-1} + 1) = p(n, P_{n-1}) \). It is impossible that \( n + 1 \leq P_{n-1} \leq 2n \); otherwise \( p(n - 1, P_{n-1} - n) = p(n, P_{n-1} - n) \). It is impossible that \( 2n + 1 \leq P_{n-1} \); otherwise \( p(n - 1, n) \geq n^2 + n + 1 \). Therefore the only possibility is \( P_{n-1} = n \).

It is impossible that \( 1 \leq P_{n-2} \leq n - 1 \); otherwise

\[ p(n - 2, P_{n-2} + 1) = p(n - 1, P_{n-2}) \]

It is impossible that \( n \leq P_{n-2} \leq 2n - 1 \); otherwise

\[ p(n - 2, P_{n-2} - n + 1) = p(n - 1, P_{n-2} - n + 1) \]

It is impossible that \( 2n \leq P_{n-2} \leq 3n - 2 \); otherwise

\[ p(n - 2, P_{n-2} - 2n + 1) = p(n - 1, P_{n-2} - 2n + 2) \]

It is impossible that \( P_{n-2} = 3n - 1 \); otherwise \( p(n - 2, n) = p(n, n) \). It is impossible that \( 3n \leq P_{n-2} \); otherwise \( p(n - 2, n) \geq n^2 + n + 1 \). If \( n \neq 2 \), then \( 1 \leq n - 2 \leq n \) and \( P_{n-2} \) is a positive integer, a contradiction. The only possibility is therefore \( n = 2 \).
From these two theorems some patterns can be seen. They are formulated in the following lemmas.

**Lemma 1.** In a perfect $n$-sequence for $m$, if $1 \leq n - r \leq m$, then

$$P_{n-r} \leq mn - n^2 + nr - r + 1.$$  

In particular, in a perfect $n$-sequence for $n + 1$, $P_{n-r} \leq nr + in - r + 1$.

**Proof.** If $P_{n-r} > mn - n^2 + nr - r + 1$, then $p(n - r, n) > mn$, which is impossible since the largest position is $mn$.

**Lemma 2.** In a perfect $n$-sequence for $m$, if $P_x$ and $P_{x+1}$ are meaningful, then it is impossible that

$$P_{x+1} + (i - 1)x + (2i - 2) \leq P_x \leq P_{x+1} + (i - 1)x + (i - 2) + n$$

for any integer $i \geq 1$, or that

$$P_{x+1} + (i - 1)x + (i - 1) \leq P_x \leq P_{x+1} + (i - 1)x + (2i - 3) + n$$

for any integer $i \leq 1$.

**Proof.** Assuming (3) to hold (with $i \geq 1$), we have

$$P_{x+1} + (i - 1)x + (2i - 2) \leq P_x$$  

(5)

$$P_x \leq P_{x+1} + (i - 1)x + (i - 2) + n.$$  

(6)

It follows from (5) and (6), respectively, that

$$P_{x+1} + (i - 1)x + (i - 1) \leq P_x$$  

(7)

$$P_x \leq P_{x+1} + (i - 1)x + (2i - 3) + n.$$  

(8)

From (5) and (8) follows

$$1 \leq P_x - P_{x+1} - ix + x - 2i + 3 \leq n,$$  

(9)

and from (7) and (6) follows

$$1 \leq P_x - P_{x+1} - ix + x - i + 2 \leq n.$$  

(10)

Finally, we have

$$p(x, P_x - P_{x+1} - ix + x - 2i + 3) = p(x + 1, P_x - P_{x+1} - ix + x - i + 2),$$  

(11)
which is meaningful by (9) and (10). But (11) is obviously false, hence (3) cannot hold if \( i \geq 1 \). The proof of the second half is identical.

**Corollary to Lemma 2.** If \( P_x \) and \( P_{x+1} \) are meaningful, then either

\[
P_{x+1} + (i - 1)x + (i - 2) + n < P_x < P_{x+1} + ix + 2i
\]

for some \( i \geq 1 \), or

\[
P_{x+1} + (i - 1)x + (2i - 3) + n < P_x < P_{x+1} + ix + i
\]

for some \( i \leq 0 \).

**Theorem 3.** There is no perfect \( n \)-sequence for \( n + 2 \), except the perfect 2-sequence for 4.

**Proof.** This sequence has \( n^2 + 2n \) entries. By Lemma 1, the only possibilities for \( P_{n+2} \) are (case I) \( P_{n+2} = 1 \), (case II) \( P_{n+2} = 2 \), and (case III) \( P_{n+2} = 3 \).

**Case I.** \( P_{n+2} = 1 \). By Lemma 1 and the Corollary to Lemma 2, the only possibilities for \( P_{n+1} \) are (case IA) \( P_{n+1} = n + 1 \) and (case IB) \( P_{n+1} = n + 2 \).

**Case IA.** \( P_{n+1} = n + 1 \). By the lemmas, the only possibilities for \( P_n \) are 1, \( n - 1 \), \( n + 1 \), and \( 2n + 1 \). But \( P_n = 1 \) is impossible; otherwise \( p(n,1) = p(n+1,1) \); \( P_n = n + 1 \) is impossible; otherwise \( p(n,1) = p(n+1,1) \). Therefore there are three possibilities: (case IA1) \( P_n = n - 1 \), (case IA2) \( P_n = n \), and (case IA3) \( P_n = 2n - 1 \).

**Case IA1.** \( P_n = n - 1 \). The possibilities for \( P_{n-1} \) are \( n - 2 \), \( 2n - 1 \), \( 3n - 1 \), and \( 3n \). But \( n \) even is impossible; otherwise \( p(n,n/2) = p(n+2,n/2) \); so \( n \) is odd; \( P_{n-1} = n - 2 \) is impossible; otherwise \( p(n-1,n+1/2) = p(n+1,n-1/2) \); \( P_{n-1} = 3n - 1 \) is impossible; otherwise \( p(n-1,n) = p(n+1,n) \); \( P_{n-1} = 3n \) is impossible; otherwise

\[
p(n - 1, (n - 1)/2) = p(n + 1, (n + 1)/2);
\]

Therefore \( P_{n-1} = 2n - 1 \). The possibilities for \( P_{n-2} \) are \( n - 1 \), \( 4n - 2 \), and \( 4n - 1 \). But \( P_{n-2} = n - 1 \) is impossible; otherwise \( p(n-2,1) = p(n,1) \); \( P_{n-2} = 4n - 2 \) is impossible; otherwise (noting that \( 1 \leq (n + 3)/2 \leq n \) since \( n \geq 2 \) and \( n \) is odd)

\[
p(n - 2, (n - 1)/2) = p(n, (n + 3)/2);
\]

\( P_{n-2} = 4n - 1 \) is impossible; otherwise \( p(n - 2,1) = p(n - 1,3) \). But \( 1 \leq n - 2 \leq n \) (since \( n \geq 2 \) and \( n \) odd) so that \( P_{n-2} \) is a positive integer, which is a contradiction. Therefore case IA1 is impossible.

This first case indicates the methods used. The others are treated similarly. The other cases are:

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