A GENERALIZED FIBONACCI SEQUENCE

P. S. FISHER
Kansas State University, Manhattan, Kansas
and
E. E. KOHLBECKER
McMurray College, Jacksonville, Illinois

Since the year 1202 when Leonardo Pisano originated the Fibonacci sequence, many interesting results have been obtained [4]. The sequence is usually defined

$$F_0 = 0 \quad F_1 = 1 \quad F_n = F_{n-1} + F_{n-2} \quad \text{for } n \geq 2 .$$

and it is a well-known fact that

$$\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^n = \begin{pmatrix} F_{n-1} & F_n \\ F_n & F_{n+1} \end{pmatrix} .$$

In this paper we generalize the usual definition of the Fibonacci numbers and the matrix relation, and exhibit some of the many relationships which hold for elements of the generalized sequence.

Consider the following definitions for a generalized sequence.

Definition 1. The k^{th} order Fibonacci sequence is a sequence which satisfies the following conditions:

a. $F_0 = F_1 = \cdots = F_{k-1} = 0$ and $F_{-i} = 0$ for all $i \geq 1$

b. $F_{k-1} = 1$

c. $F_n = \sum_{i=1}^{k} F_{n-i}$ for $n \geq k$.

If we relax the condition as specified in part a of Definition 1, we obtain the following:

Definition 2. A sequence whose members (denoted $\overline{F}_i$) satisfy the following two conditions will be called a generalized k^{th} order Fibonacci sequence.

a. $\overline{F}_i = a_i$ for $0 \leq i \leq k - 1$ where a_i is an arbitrary number,

b. $F_n = \sum_{i=1}^{k} \overline{F}_{n-1}$.

We now define a sequence called the r^{th} auxiliary sequence of order k as a special case of the generalized k^{th} order sequence.

Definition 3. A sequence which satisfies the following three conditions will be called an r^{th} auxiliary sequence of order k, where $1 \le r \le k - 2$.

a. $A_i^r = 0$ for $0 \le i \le k - 2$, $i \ne r - 1$

b. $A_{r-1}^r = A_{k-1}^r = 1$

c. $A_n^r = \sum_{i=1}^{k} A_{n-i}^r$ for $n \ge k$.

In the following, the superscript of F_i^k will be left off if it is clear from the context that we are concerned with the k^{th} order sequence.

Property 1. $F_j^k = F_{j-1}^{k-1}$ for $1 \le j \le 2(k - 1)$ $k > 2$

Property 2. $F_{2k-1}^k = F_{2(k-1)}^{k-1} + 1$ for all $k > 1$

Property 3. $F_{2k}^k = 2^k - 1$ for all $k \ge 2$.

Theorem 1. If A_n^r is an element in the r^{th} auxiliary sequence of order k, and if F_i is an element of the corresponding k^{th} order Fibonacci sequence, then

$$A_n^r = F_n + F_{n-1} + \cdots + F_{n-r+1} + F_{n-r} \quad \text{for } n \ge k.$$

Proof. Let $n = k$, then we will show that

$$A_k^r = F_k + F_{k-1} + \cdots + F_{k-r} \quad . \tag{1}$$

By Definition 1,

$$\sum_{i=0}^{r} F_{k-i} = 2 \quad , \tag{2}$$

and by Definition 2,

$$A_k^r = \sum_{j=1}^{k} A_{k-j}^r \quad . \tag{3}$$

But since r is defined in the range $0 \le r \le k - 2$ then there is an element A_{k-j}^r in (3) such that $k - j = r - 1$. From the definition of A_i^r we know

$$A_{r-1}^r = A_{k-1}^r = 1 \quad ,$$

and all the remaining elements are zero. Therefore

$$\sum_{j=1}^{k} A^r_{j-j} = 2$$

and from (2) we have the desired result for $n = k$.

Suppose that for $k \le n \le m$ the theorem is true, then for $n = m + 1$, we will also show the theorem is true.

$$A^r_{m+1} = \sum_{j=1}^{k} A^r_{m+1-j} = A^r_m + A^r_{m-1} + \cdots + A^r_{m+1-k} \tag{4}$$

By hypothesis we can rewrite each element of (4) as follows:

$$\begin{aligned} A^r_m &= F_m + F_{m-1} + \cdots + F_{m-r} \\ A^r_{m-1} &= F_{m-1} + F_{m-2} + \cdots + F_{m-r-1} \\ &\vdots \\ A^r_{m+1-k} &= F_{m+1-k} + F_{m-k} + \cdots + F_{m+1-k-r} \end{aligned}$$

and adding the columns we obtain

$$\begin{aligned} \sum_{i=1}^{k} A^r_{m-i+1} &= \sum_{j=1}^{k} F_{m-j+1} + \sum_{j=1}^{k} F_{m-j} + \cdots + \sum_{j=1}^{k} F_{m-j-r+1} \\ &= F_{m+1} + F_m + \cdots + F_{m-r+1} \\ &= A^r_{m+1} \end{aligned}$$

which is the desired result.

Lemma 1. The following three identities hold for elements of the auxiliary sequences for $m \ge k$ and $1 \le r \le k - 2$.

A. $$A^r_m - A^{r-1}_{m-1} = F_m$$

B. $$A^r_m - A^{r-1}_m = F_{m-r}$$

C. $$A^r_m - A^r_{m-1} = -F_{m-r-1} + F_m$$

Theorem 2. If Q is the $k \times k$ matrix

$$\begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 & 0 & 0 \\ & \vdots & & & & & \vdots & \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & \cdots & 1 & 1 & 1 \end{bmatrix},$$

then

$$Q^n = \begin{bmatrix} F_{n-1} & A^1_{n-1} & \cdots & A^r_{n-1} & \cdots & A^{k-2}_{n-1} & F_n \\ F_n & A^1_n & \cdots & A^r_n & \cdots & A^{k-2}_n & F_{n+1} \\ F_{n+1} & A^1_{n+1} & \cdots & A^r_{n+1} & \cdots & A^{k-2}_{n+1} & F_{n+2} \\ \vdots & \vdots & & \vdots & & \vdots & \vdots \\ F_{n+k-4} & A^1_{n+k-4} & \cdots & A^r_{n+k-4} & \cdots & A^{k-2}_{n+k-4} & F_{n+k-3} \\ F_{n+k-3} & A^1_{n+k-3} & \cdots & A^r_{n+k-3} & \cdots & A^{k-2}_{n+k-3} & F_{n+k-2} \\ F_{n+k-2} & A^1_{n+k-2} & \cdots & A^r_{n+k-2} & \cdots & A^{k-2}_{n+k-2} & F_{n+k-1} \end{bmatrix}$$

where n is a positive integer, and the F_n's are elements of the k^{th} order sequence and the A^r_{n+1} are the corresponding terms of the r^{th} auxiliary sequence of that same order.

Proof. This theorem can be proved by induction on n. With $n = 2$, Q^2 is

$$Q^2 = \begin{bmatrix} 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 & 0 \\ & & \vdots & & & & \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & 2 & 2 & 2 & \cdots & 2 & 2 \end{bmatrix} = \begin{bmatrix} F_1 & A^1_1 & \cdots & A^{k-2} & F_2 \\ F_2 & A^1_2 & \cdots & A^{k-2}_2 & F_3 \\ \vdots & & & & \\ F_{k-2} & A^1_{k-2} & \cdots & A^{k-2}_{k-2} & F_{k-1} \\ F_{k-1} & A^1_{k-1} & \cdots & A^{k-2}_{k-1} & F_k \\ F_k & A^1_k & \cdots & A^{k-2}_k & F_{k+1} \end{bmatrix}$$

Supposing the theorem is true for $1 \le n \le m$ we can show it is true for $n = m + 1$.

$$Q^{m+1} = Q \cdot Q^m \; .$$

But examining the effect of multiplying Q with Q^m, it is obvious that the first $k - 1$ rows of Q cause row i $(2 \le i \le k)$ to become row $i - 1$ of Q^{m+1}. The k^{th} row of Q^{m+1} is obtained by summing the columns of Q^m, which using definitions 1 and 3 produces the desired result.

Theorem 3. If $n = k$, then

$$\sum_{i=1}^{n} F_i = -F_{n+k+1} - \frac{1}{k-1} + \sum_{i=1}^{k} iF_{n+i}$$

or

$$\sum_{i=1}^{n} F_i = -\frac{1}{k-1} + \sum_{i=1}^{k} (i-1)F_{n+i} \; .$$

Proof. The sum of the first n terms of the k^{th} order sequence will appear as an element in a matrix which represents the sum

$$\sum_{i=1}^{n} Q^i \tag{1}$$

in either the $(1,k)$ or the $(2,1)$ position. We rewrite (1) to obtain

$$\sum_{i=1}^{n} Q^i = (Q - I)^{-1} Q(Q^n - I) \tag{2}$$

where I is the $(k \times k)$ identity matrix. The inverse of $Q - I$ shown in (2) can be shown to be

$$\frac{1}{k-1}\begin{bmatrix} -(k-2) & -(k-3) & \cdots & -(k-k) & -(k-(k+1)) \\ 1 & -(k-3) & \cdots & -(k-k) & -(k-(k+1)) \\ 1 & 2 & \cdots & -(k-k) & -(k-(k+1)) \\ \vdots & \vdots & & \vdots & \vdots \\ 1 & 2 & \cdots & -(k-k) & -(k-(k+1)) \\ 1 & 2 & \cdots & k-1 & -(k-(k+1)) \end{bmatrix} \tag{3}$$

If we multiply the first row of (3) against the last column of $Q^{n+1} - Q$ we obtain the element in the $(1,k)$ position which represents the desired sum

$$\sum_{i=1}^{n} F_i \ .$$

$$\begin{aligned} \sum_{i=1}^{n} F_i &= \frac{1}{k-1}\sum_{i=1}^{k} - (k - i - 1)(F_{n+i} - F_i) \\ &= \frac{1}{k-1}\left[-(k-1)(F_{n+k+1} - F_{k+1}) + \sum_{i=1}^{k} i(F_{n+i} - F_i)\right] \\ &= F_{n+k+1} + 2 + \frac{1}{k-1}\sum_{i=1}^{k} iF_{n+1} - \frac{k}{k-1}\sum_{i=1}^{k} iF_i \\ &= -F_{n+k+1} - \frac{1}{k-1} + \sum_{i=1}^{k} i\,F_{n+i} \end{aligned} \tag{4}$$

which is the desired result. The second form can be obtained by substituting in (4) the sum

$$\sum_{i=1}^{k} F_{n+i}$$

for F_{n-k+1}, and combining the two sums.

Using the method given in the proof of Theorem 3, it is obvious that an expression similar to that given in this theorem can be given for the sum

$$\sum_{i=1}^{n} F_{i+j-1} \quad \text{for} \quad 1 \le j - 1 < k \quad \text{and} \quad n \ge k$$

which is

$$\sum_{i=1}^{n} F_{i+j-1} = \frac{1}{k-1}\sum_{i=1}^{k} i\,F_{n+i} + \frac{(zk-1)(k-2)}{k-1} - \sum_{i=j}^{k} F_{n+i}$$

Theorem 4. If $0 \le j \le k - 1$ and if $n \ge k$ then

$$\sum_{i=1}^{n} F_{ki+j} = \frac{1}{k-1}\sum_{i=1}^{k} i\,F_{nk+k-1} - \sum_{i=j+1}^{k} F_{nk+i-1} - \frac{1}{k-1} .$$

Proof. Consider the sum

(1) $$\sum_{i=1}^{n} Q^{ki}$$

We can obtain the desired sum

$$\sum_{i=1}^{n} F_{ki+j}$$

in the (k,j) position of the matrix representing the sum given in (1).

(2) $$\begin{aligned}\sum_{i=1}^{n} Q^{ki} &= Q^k\,[I + Q^k + \cdots + Q^{(n-1)k}] \\ &= (Q^k - I)^{-1} Q^k (Q^{nk} - 1)\end{aligned}$$

The characteristic equation of Q is

$$x^k - x^{k-1} - x^{k-2} - \cdots - x - 1$$

and since Q^{k-1} always has a factor $Q - I$, we can write

$$Q^{k-1} = (Q - I)(Q^{k-1} + Q^{k-2} + \cdots + Q + I) .$$

However

$$Q^{k-1} + Q^{k-2} + \cdots + Q + I = Q^k$$

therefore

$$Q^k - 1 = Q^k(Q - I)$$

and thus

$$(Q^k - I)^{-1}Q^k(Q^{nk} - I) = (Q - I)^{-1}(Q^{nk} - I) \tag{3}$$

and upon multiplying the last column of $Q^{nk} - I$ with the j^{th} row of $(Q - I)^{-1}$ we obtain the desired result.

Theorem 5.

$$F_{m+n} = F_{m-1}F_n + F_m F_{n+k-1} + \sum_{i=1}^{k-2} A^i_{m-1} F_{n+i} ,$$

where m and n are nonnegative integers.

Note. If $k = 2$ then $F_{m+n} = F_{m-1}F_n + F_{n+1}F_m$ which is a well-known result of the usual Fibonacci sequence.

Proof. F_{m+n} occurs in the matrix Q^{m+n} in the $(2,1)$ or $(1,k)$ positions. Since

$$Q^{m+n} = Q^m Q^n ,$$

the required multiplication can be performed to yield the desired result.

Theorem 6. If B is the matrix

$$\begin{bmatrix} \overline{F}_0 & A^1_0 & \cdots & A^r_0 & \cdots & A^{k-2}_0 & \overline{F}_1 \\ \overline{F}_1 & A^1_1 & \cdots & A^r_1 & \cdots & A^{k-2}_1 & \overline{F}_2 \\ \overline{F}_2 & A^1_2 & \cdots & A^r_2 & \cdots & A^{k-2}_2 & \overline{F}_3 \\ \vdots & \vdots & & \vdots & & \vdots & \vdots \\ \overline{F}_{k-2} & A^1_{k-2} & \cdots & A^r_{k-2} & \cdots & A^{k-2}_{k-2} & \overline{F}_{k-1} \\ \overline{F}_{k-1} & A^1_{k-1} & \cdots & A^r_{k-1} & \cdots & A^{k-2}_{k-2} & \overline{F}_k \end{bmatrix}$$

where the A_i^r, $(1 \le r \le k - 2)$ are the elements of the r^{th} auxiliary sequence of order k, and if Q^{-n} is the matrix

$$\begin{bmatrix} \overline{F}_{n-1} & \cdots & A_{n-1}^r & \cdots & \overline{F}_n \\ \overline{F}_n & \cdots & A_n^r & \cdots & \overline{F}_{n+1} \\ \vdots & & \vdots & & \vdots \\ \overline{F}_{n+k-3} & \cdots & A_{n+k-3}^r & \cdots & \overline{F}_{n+k-2} \\ \overline{F}_{n+k-2} & \cdots & A_{n+k-2}^r & \cdots & \overline{F}_{n+k-1} \end{bmatrix}$$

then $Q^{n-1}B = Q^{-n}$ where Q is the matrix defined in Theorem 2.

Proof. The proof is again done by induction upon n and is similar to that given in Theorem 2.

Using the results of Theorem 6 for the generalized Fibonacci sequence, it is possible to obtain theorems for this sequence corresponding to Theorems 3, 4, 5. These corresponding theorems are stated here without proof.

Theorem 7.

$$\sum_{i=1}^{n} \overline{F}_i = -\overline{F}_{n+k+1} + \overline{F}_{k+1} + \frac{1}{k-1} \sum_{i=1}^{k} i(\overline{F}_{n+1} - \overline{F}_i) \quad \text{for } n \ge k.$$

Theorem 8. If $1 \le j \le k - 1$, and if $n \ge k$, then

$$\sum_{i=1}^{n} \overline{F}_{ik+j} = \frac{1}{k-1} \sum_{i=1}^{k} i(\overline{F}_{nk+i} - \overline{F}_i) + \sum_{i=j}^{k} (\overline{F}_{nk+i} - \overline{F}_i)$$

and with $j = 0$, the expression becomes

$$\sum_{i=1}^{n} \overline{F}_{ki} = \frac{1}{k-1} \left[\sum_{i=1}^{k-1} i(\overline{F}_{nk+i} - \overline{F}_i) - (k-2)(\overline{F}_{(n+1)k} - \overline{F}_k) \right]$$

Theorem 9.

$$\overline{F}_{m+n} = \overline{F}_{m-1}\overline{F}_n + \overline{F}_m \overline{F}_{n+k+1} + \sum_{i=1}^{k-2} A_{m-1}^i \overline{F}_{n+i} \quad \text{for } m,n \ge 1 .$$

There are many known relations involving the elements of the usual or 2^{nd} order Fibonacci sequence. A partial list of these relations appear in [4]. However, when generalizing many of these relations to the terms of either the k^{th} order or the generalized k^{th} order sequence, the relations become quite involved; but in each case a corresponding formula holds in the more general situation.

[Continued on page 354.]