A DISTRIBUTION PROPERTY OF THE SEQUENCE OF FIBONACCI NUMBERS

LAWRENCE KUIPERS and JAU-SHYONG SHIUE
Southern Illinois University, Carbondale, Illinois

Let $\{F_n\}$ $(n = 1, 2, \cdots)$ be the Fibonacci sequence. Then in order to prove the main theorems of this paper we need the following lemmas (see [2]).

Lemma 1. Every Fibonacci number F_k divides every Fibonacci number F_{nk} for $n = 1, 2, \cdots$.

Lemma 2. $(F_m, F_n) = F_{(m,n)}$ where (x,y) denotes the greatest common divisor of the integers x and y.

Lemma 3. Every positive integer m divides some Fibonacci number whose index does not exceed m^2.

Lemma 4. Let p be an odd prime and $\neq 5$. Then p does not divide F_p.

Proof of Lemma 4. According to [1], p. 394, we have that either F_{p-1} or F_{p+1} is divisible by p. From the well known identity $F_{n+1}F_{n-1} - F_n^2 = (-1)^n$, we derive that $p \nmid F_p$.

Definition 1. The sequence of integers $\{x_n\}$ $(n = 1, 2, \cdots)$ is said to be uniformly distributed mod m where $m \geq 2$ is an integer, provided that

$$\lim_{N\to\infty} \frac{1}{N} \cdot A(N, j, m) = \frac{1}{m} \quad ,$$

for each j, $j = 0, 1, \cdots, m-1$, where $A(N, j, m)$ is the number of x_n, $n = 1, 2, \cdots, N$, that are congruent to $j \pmod{m}$.

Theorem 1. Let $\{F_n\}$ $(n = 1, 2, \cdots)$ be the Fibonacci sequence. Then $\{F_n\}$ is uniformly distributed mod 5.

Proof. Let all F_n $(n = 1, 2, \cdots)$ be reduced mod 5. Then we obtain the following sequence of least residues:

1, 1, 2, 3, 0, 3, 3, 1, 4, 0, 4, 4, 3, 2, 0, 2, 2, 4, 1, 0, 1, 1, 2, 3, , ···

Obviously, this sequence is periodic with the period length 20. Now evidently

$$\lim_{N\to\infty} \frac{1}{N} \cdot A(N, j, 5) = \frac{1}{5} \quad \text{for} \quad j = 0, 1, 2, 3, 4 \ ,$$

or, $\{F_n\}$ is uniformly distributed mod 5.

Theorem 2. Let $\{F_n\}$ $(n = 1, 2, \cdots)$ be the Fibonacci sequence. Then $\{F_n\}$ is not uniformly distributed mod 2.

Proof. This follows from the fact that the sequence of least residues of $\{F_n\}$ is 1, 1, 0, 1, 1, 0, ···.

Theorem 3. Let $\{F_n\}$ $(n = 1, 2, \cdots)$ be the Fibonacci sequence. Then $\{F_n\}$ is not uniformly distributed mod p for any prime $p > 2$ and $\neq 5$.

Proof. Let p be a prime > 2 and $\neq 5$. Because of Lemmas 3 and 4 there exists a positive integer $t \neq p$ such that $F_t \equiv 0 \pmod p$. We may suppose that t is the smallest positive integer with this property. By Lemma 1, we have $F_{kt} \equiv 0 \pmod p$ for $k = 1, 2, \cdots$. Now there does not exist a positive integer q with $kt < q < (k+1)t$ $(k = 1, 2, \cdots)$ such that $F_q \equiv 0 \pmod p$, for otherwise there would exist an r $(0 < r < t)$ with $F_r \equiv 0 \pmod p$, which can be seen as follows. Let there be a q with the aforementioned property, then by virtue of Lemma 2, we would have

$$(F_{kt}, F_q) = F_{(kt,q)} \equiv 0 \pmod p .$$

Now write $q = kt + r$ $(0 < r < t)$ and therefore

$$(kt, q) = (kt, kt + r) = (kt, r) \leq r < t .$$

Because of the above property of t we have that

$$A(N, 0, p) = \left[\frac{N}{t}\right] ,$$

where [a] denotes the integral part of a, and A(N, 0, p) is related to the Fibonacci sequence (see Definition 1). Let

$$N = \left[\frac{N}{t}\right] t + r$$

with $0 \leq r < t$. Then

$$A(N, 0, p) = \frac{N - r}{t} ,$$

and therefore

$$\frac{1}{N} \cdot A(N, 0, p) = \frac{1}{t} - \frac{r}{Nt} ,$$

so

$$\lim_{N \to \infty} \frac{1}{N} \cdot A(N, 0, p) = \frac{1}{t} \qquad (t \neq p)$$

for any prime $p > 2$ and $\neq 5$. Hence $\{F_n\}$ is not uniformly distributed mod p for any prime $p > 2$ and $\neq 5$.

Theorem 4. Let $\{F_n\}$ $(n = 1, 2, \cdots)$ be the Fibonacci sequence. Then $\{F_n\}$ is not uniformly distributed mod m for any composite integer $m > 2$ and $m \neq 5^k$ $(k = 3, 4, \cdots)$.

Proof. Suppose that $\{F_n\}$ is uniformly distributed mod m for some composite integer m as indicated in the theorem. According to a theorem of I. Niven [3], Theorem 5.1,

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