A DISTRIBUTION PROPERTY OF THE SEQUENCE OF FIBONACCI NUMBERS

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Let \( \{F_n\} \) \((n = 1, 2, \cdots)\) be the Fibonacci sequence. Then in order to prove the main theorems of this paper we need the following lemmas (see [2]).

**Lemma 1.** Every Fibonacci number \( F_k \) divides every Fibonacci number \( F_{nk} \) for \( n = 1, 2, \cdots \).

**Lemma 2.** \( (F_m, F_n) = F_{(m,n)} \) where \( (x,y) \) denotes the greatest common divisor of the integers \( x \) and \( y \).

**Lemma 3.** Every positive integer \( m \) divides some Fibonacci number whose index does not exceed \( m^2 \).

**Lemma 4.** Let \( p \) be an odd prime and \( p \neq 5 \). Then \( p \) does not divide \( F_p \).

**Proof of Lemma 4.** According to [1], p. 394, we have that either \( F_{p-1} \) or \( F_{p+1} \) is divisible by \( p \). From the well known identity \( F_{n+1} F_{n-1} - F_n^2 = (-1)^n \), we derive that \( p \nmid F_p \).

**Definition 1.** The sequence of integers \( \{x_n\} \) \((n = 1, 2, \cdots)\) is said to be uniformly distributed mod \( m \) where \( m \geq 2 \) is an integer, provided that

\[
\lim_{N \to \infty} \frac{1}{N} \cdot A(N, j, m) = \frac{1}{m}
\]

for each \( j, j = 0, 1, \cdots, m-1 \), where \( A(N, j, m) \) is the number of \( x_n \), \( n = 1, 2, \cdots, N \), that are congruent to \( j \) (mod \( m \)).

**Theorem 1.** Let \( \{F_n\} \) \((n = 1, 2, \cdots)\) be the Fibonacci sequence. Then \( \{F_n\} \) is uniformly distributed mod 5.

**Proof.** Let all \( F_n \) \((n = 1, 2, \cdots)\) be reduced mod 5. Then we obtain the following sequence of least residues:

\[1, 1, 2, 3, 0, 3, 1, 4, 0, 4, 0, 3, 2, 2, 0, 2, 2, 4, 1, 0, 1, 1, 2, 3, \cdots\]

Obviously, this sequence is periodic with the period length 20. Now evidently

\[
\lim_{N \to \infty} \frac{1}{N} \cdot A(N, j, 5) = \frac{1}{5}
\]

for \( j = 0, 1, 2, 3, 4 \).

or, \( \{F_n\} \) is uniformly distributed mod 5.

**Theorem 2.** Let \( \{F_n\} \) \((n = 1, 2, \cdots)\) be the Fibonacci sequence. Then \( \{F_n\} \) is not uniformly distributed mod 2.

**Proof.** This follows from the fact that the sequence of least residues of \( \{F_n\} \) is \( 1, 1, 0, 1, 1, 0, \cdots \).
Theorem 3. Let \( \{F_n\} \) \((n = 1, 2, \cdots)\) be the Fibonacci sequence. Then \( \{F_n\} \) is not uniformly distributed mod \( p \) for any prime \( p > 2 \) and \( \neq 5 \).

Proof. Let \( p \) be a prime \( >2 \) and \( \neq 5 \). Because of Lemmas 3 and 4 there exists a positive integer \( t \neq p \) such that \( F_t \equiv 0 \pmod{p} \). We may suppose that \( t \) is the smallest positive integer with this property. By Lemma 1, we have \( F_{kt} \equiv 0 \pmod{p} \) for \( k = 1, 2, \cdots \). Now there does not exist a positive integer \( q \) with \( kt < q < (k + 1)t \) \((k = 1, 2, \cdots)\) such that \( F_q \equiv 0 \pmod{p} \), for otherwise there would exist an \( r \) \((0 < r < t)\) with \( F_r \equiv 0 \pmod{p} \), which can be seen as follows. Let there be a \( q \) with the aforementioned property, then by virtue of Lemma 2, we would have

\[
(F_{kt}, F_q) = F_{(kt, q)} = 0 \pmod{p}.
\]

Now write \( q = kt + r \) \((0 < r < t)\) and therefore

\[
(kt, q) = (kt, kt + r) = (kt, r) \leq r < t.
\]

Because of the above property of \( t \) we have that

\[
A(N, 0, p) = \left(\frac{N}{t}\right),
\]

where \([a]\) denotes the integral part of \( a \), and \( A(N, 0, p) \) is related to the Fibonacci sequence (see Definition 1). Let

\[
N = \left[\frac{N}{t}\right] t + r
\]

with \( 0 \leq r < t \). Then

\[
A(N, 0, p) = \frac{N - r}{t},
\]

and therefore

\[
\frac{1}{N} \cdot A(N, 0, p) = \frac{1}{t} - \frac{r}{Nt},
\]

so

\[
\lim_{N \to \infty} \frac{1}{N} \cdot A(N, 0, p) = \frac{1}{t} \quad (t \neq p)
\]

for any prime \( p \geq 2 \) and \( \neq 5 \). Hence \( \{F_n\} \) is not uniformly distributed mod \( p \) for any prime \( p > 2 \) and \( \neq 5 \).

Theorem 4. Let \( \{F_n\} \) \((n = 1, 2, \cdots)\) be the Fibonacci sequence. Then \( \{F_n\} \) is not uniformly distributed mod \( m \) for any composite integer \( m > 2 \) and \( m \neq 5^k \) \((k = 3, 4, \cdots)\).

Proof. Suppose that \( \{F_n\} \) is uniformly distributed mod \( m \) for some composite integer \( m \) as indicated in the theorem. According to a theorem of I. Niven [3], Theorem 5.1, [Continued on page 392.]