INTRODUCTION TO PATTON POLYGONS

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This paper introduces an extraordinarily elementary topic which is accessible to any patient high school student with little or no sophisticated number theory. The ideas covered are presented in a straight-forward fashion, with many proofs and extensions left for the reader to work through. Deeper connexions with additive sequences and number theory are left to those with interest to pursue matters in the standard references on Fibonacci numbers. In the following (*) designates assertions which must be proved or developed by the reader. Drawing all the figures carefully is certainly essential to an understanding of what is going on.

1. Choose a coordinate system (which is to say, use some convenient graph paper) and draw any parallelogram $0A_0A_2A_1$ where 0 is the origin, and the letters are taken around the figure.

2. Find the unique point A_3 so that $0A_1A_3A_2$ is a parallelogram (Fig. 1).

3. In general, find the point A_{n+1} so that $0A_{n-1}A_{n+1}A_n$ is a parallelogram.

*4. Consider the situation if $n = -1, -2, -3, \cdots$ in (3) and study Fig. 2.

5. If we have been successful so far, we now have a set of points $\{A_n\}$ where n is any integer, positive or negative; we may consider these points as forming an infinite polygon $\cdots A_{-2}A_{-1}A_0A_1A_2A_3\cdots$ (Fig. 3).

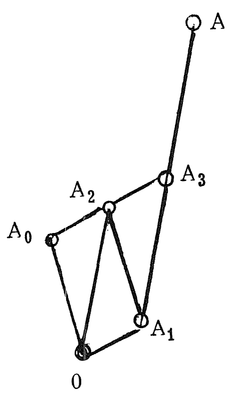

Figure 1

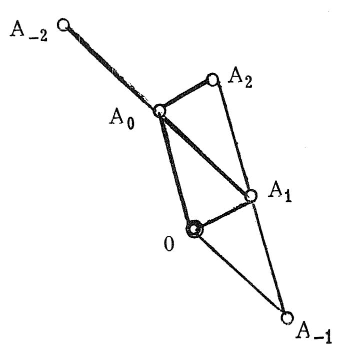

Figure 2

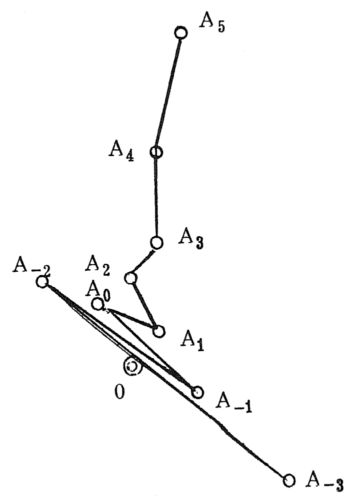

Figure 3

6. This curious polygon has many properties which are somewhat surprising. Evidently, A_{n+2} is the midpoint of $\overline{A_n A_{n+3}}$ for any integer n. This can be easily shown since $A_0A_2 = 0A_1 = A_2A_3$ as opposite sides in the first two parallelograms. This process may be continued along the polygon.

7. But there is a more interesting and related result. In Fig. 1, area $0A_1A_2 = \frac{1}{2}$ area $0A_1A_2A_0$ = area $A_0A_1A_2$, and $A_0A_2 = A_2A_3$, so that area $A_0A_1A_2$ = area $A_1A_2A_3$. Continuing along the polygon we find that area $A_1A_2A_3$ = area $A_2A_3A_4$. In general then, area $0A_0A_1$ = area $A_nA_{n+1}A_{n+2}$. In a sense, the polygon is an infinite stack of triangles with the same area.

8. Vectors are now introduced to make calculations a bit simpler. Let $\overrightarrow{0A_n}$ be represented by the vector v_n. We may apply the "Parallelogram Law" for vector addition to $0A_0A_1A_2$ so that we have $\overrightarrow{0A_0} + \overrightarrow{0A_1} = \overrightarrow{0A_2}$, or $v_0 + v_1 = v_2$. In general, we have that $v_{n+2} = v_{n+1} + v_n$, since by (3), $0A_nA_{n+2}A_{n+1}$ is a parallelogram.

9. The entire polygon is based on $0A_0A_2A_1$, so in some way, the vectors v_0 and v_1 are fundamental. In fact,

$$\begin{aligned} v_2 &= v_1 + v_0 \\ v_3 &= v_2 + v_1 = 2v_1 + v_0 \\ v_4 &= v_3 + v_2 = 3v_1 + 2v_0 \\ v_5 &= v_4 + v_3 = 5v_1 + 3v_0 \\ v_6 &= v_5 + v_4 = 8V_1 + 5v_0 \ . \end{aligned}$$

And we recognize our old friend the Fibonacci sequence where $F_0 = 0$, $F_1 = 1$, and $F_{n+2} = F_{n+1} + F_n$. In short, we are able to write: $v_n = F_nv_1 + F_{n-1}v_0$.

*10. In the negative direction along the polygon, check that $v_{-n} = F_{-n}v_1 + F_{-n-1}v_0$. We already know one of the properties of the Fibonacci sequence is that

$$F_{-n} = (-1)^{n+1}F_n ,$$

and so we have $v_{-n} = (-1)^{n+1}(F_nv_1 - F_{n+1}v_0)$.

11. Using the coordinate system we set up in (1), we may assign coordinates (f_n, g_n) to the point A_n; and, of course, the vector v_n will have the same coordinates. Then, since vectors are added coordinate-wise, we have:

$$f_n = F_nf_1 + F_{n-1}f_0 \qquad \text{and} \qquad g_n = F_ng_1 + F_{n-1}g_0 ,$$

for any integer n.

*12. Since our polygons seem to be deeply involved with the Fibonacci sequence, we need a short detour to pick up some well known properties of this sequence. Let

$$\varphi = \tfrac{1}{2}(1 + \sqrt{5}) ,$$

so that $\varphi^2 = \varphi + 1$. Then F_{n+1}/F_n is an increasing sequence of rational numbers bounded by φ if n is odd, and a decreasing sequence bounded by φ if n is even. As n becomes large, F_{n+1}/F_n can be shown to approach φ as limit. As a result of all this we can write that:

$$\lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \varphi \;;$$

and that $F_n - \varphi F_{n-1} > 0$ and $\varphi F_n - F_{n+1} > 0$ if and only if n is odd.

13. Returning to the polygon, consider the <u>slope</u> of $\overline{0A_n}$ for large positive n, where A_n is the point (f_n, g_n):

$$\frac{g_n - 0}{f_n - 0} = \frac{F_n g_1 + F_{n-1} g_0}{F_n f_1 + F_{n-1} f_0} = \frac{\dfrac{F_n}{F_{n-1}} g_1 + g_0}{\dfrac{F_n}{F_{n-1}} f_1 + f_0} \;.$$

As n becomes very large, F_n/F_{n-1} approaches φ and so the slope approaches the value

$$M = \frac{\varphi g_1 + g_0}{\varphi f_1 + f_0} \;.$$

14. For the slope of $\overline{0A_{-n}}$, we find, using (10), that:

$$\frac{g_{-n} - 0}{f_{-n} - 0} = \frac{(-1)^{n+1}(F_n g_1 - F_{n+1} g_0)}{(-1)(F_n g_1 - F_{n+1} f_0)} = \frac{g_1 - \dfrac{F_{n+1}}{F_n} g_0}{f_1 - \dfrac{F_{n+1}}{F_n} f_0} \;.$$

Again, as n becomes large, the slope approaches

$$N = \frac{g_1 - \varphi g_0}{f_1 - \varphi f_0} \;.$$

15. Another way of thinking about (13) and (14) is to call the lines $y = Mx$ and $y = Nx$ the <u>asymptotes</u> of the polygon (Fig. 4), where M and N are given in (13) and (14). For large n, the polygon runs along the asymptote $y = Mx$ in the positive direction, and along $y = Nx$ in the negative direction

*16. It is easy to show that the asymptotes are distinct lines through the origin 0. Merely show that $M \neq N$ if $0A_0A_1$ form a triangle.

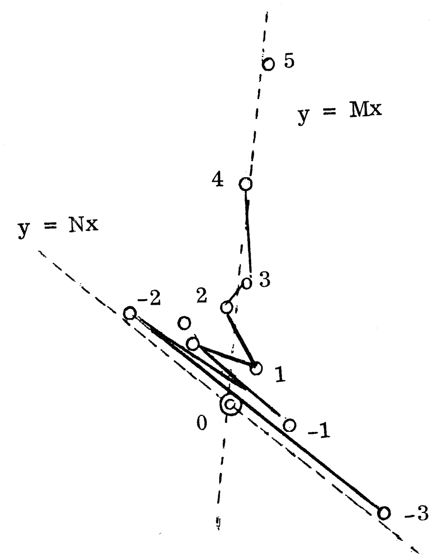

Figure 4

17. Figure 4 suggests a more intriguing relationship between the polygon and its asymptotes. In order to get at this, let

$$d = f_0g_1 - f_1g_0 = \begin{vmatrix} f_0 & g_0 \\ f_1 & g_1 \end{vmatrix}.$$

Check to see that we may choose A_0 and A_1 so that $d \neq 0$, $\varphi f_1 + f_0 \neq 0$ and $f_1 - \varphi f_0 \neq 0$. A calculation shows that for positive n:

$$g_n - Mf_n = \frac{d(F_n - \varphi F_{n-1})}{\varphi f_1 + f_0} .$$

Since $d/(\varphi f_1 + f_0)$ is a constant, the sign of $g_n - Mf_n$ depends on $F_n - \varphi F_{n-1}$ which is positive if n is odd and negative if n is even (see (12)). Hence $g_n - Mf_n$ is alternately greater and less than 0, which is equivalent to saying that g_n is alternately greater and less than Mf_n. Hence, the vertices $A_n = (f_n, g_n)$ lie alternately above and below the line $y = Mx$.

*18. A similar analysis for $g_n - Nf_n$, $g_{-n} - Mf_{-n}$ and $g_{-n} - Nf_{-n}$ yields this result: the polygon (its vertices, at any rate) lies on alternate sides of the asymptote $y = Mx$, and entirely on one side of $y = Nx$. This explains the "T"-shape of the polygon (Fig. 4). The asymptotes divide the plane into 4 regions: one containing the even-numbered vertices, another the odd ones, and the last two regions are empty.

19. We know from (7) that the absolute value of the area of triangle $A_nA_{n+1}A_{n+2}$ equals area $0A_0A_1$. More precisely, from analytic geometry, the area $0A_0A_1$ is given by the <u>determinant:</u>

$$\frac{1}{2}\begin{vmatrix} 1 & 0 & 0 \\ 1 & f_0 & g_0 \\ 1 & f_1 & g_1 \end{vmatrix},$$

which gives after expansion: $\frac{1}{2}(f_0g_1 - f_1g_0) = \frac{1}{2}d$, as in (17). Using determinants to find area, we must recall that lettering a triangle in the opposite sense changes the sign of its area. Hence we get:

$$d = \begin{vmatrix} 1 & 0 & 0 \\ 1 & f_0 & g_0 \\ 1 & f_1 & g_1 \end{vmatrix} = -\begin{vmatrix} 1 & f_0 & g_0 \\ 1 & f_1 & g_1 \\ 1 & f_2 & g_2 \end{vmatrix},$$

which is twice the area $A_0A_1A_2$, and, in general:

$$d = (-1)^{n+1}\begin{vmatrix} 1 & f_n & g_n \\ 1 & f_{n+1} & g_{n+1} \\ 1 & f_{n+2} & g_{n+2} \end{vmatrix}.$$

This in turn may be simplified to:

$$d = \begin{vmatrix} f_0 & g_0 \\ f_1 & g_1 \end{vmatrix} = (-1)^n\begin{vmatrix} f_n & g_n \\ f_{n+1} & g_{n+1} \end{vmatrix}$$

for any n. This is a rather simple and unexpected result.

*20. A little more digging around can give us even more curious results. For example, confine attention to the even-numbered vertices. These form an "hyperbola"-shaped polygon with the obvious asymptotes (Fig. 5). It can be shown without much trouble that

$$\frac{1}{2}d = \text{area } 0A_{2n}A_{2n+2} = \text{area } A_{2n}A_{2n+2}A_{2n+4}$$

in absolute value. Notice also that $F_{n+4} = 3F_{n+2} - F_n$.

*21. Check the situation for the odd-numbered vertices.

22. What happens if we demand the asymptotes be perpendicular? Borrowing a result from analytic geometry again, we see that $MN = -1$ in that case. This can be simplified to:

$$\frac{g_1^2 - g_0g_2}{f_1^2 - f_0f_2} = -1 .$$

A simple way (not the only way, of course) for this to happen is for $g_n = f_{n-1}$. This gives us the polygon with vertices (f_n, f_{n-1}) and the asymptotes are $y = (1/\varphi)x$ and $y = -\varphi x$ (which are clearly perpendicular).

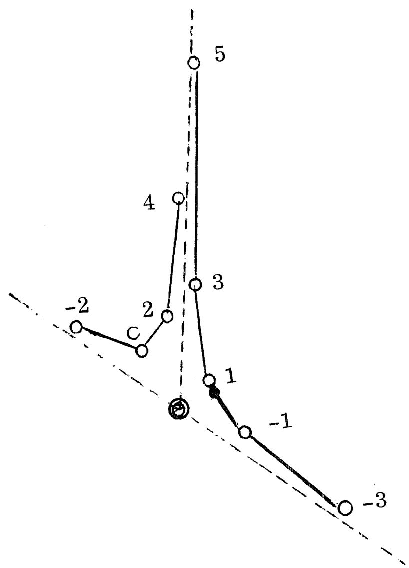

Figure 5

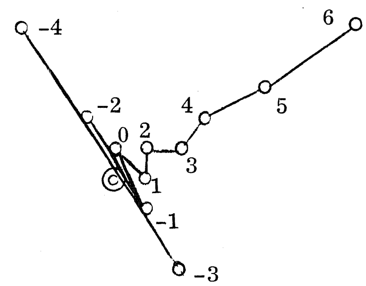

Figure 6

23. All polygons of the form (f_n, f_{n-1}) have the same asymptotes and so must be of the same general shape. The simplest one is (F_n, F_{n-1}) so that $A_0 = (0,1)$, $A_1 = (1,0)$ and $A_2 = (1,1)$ as in Fig. 6. Thus the polygon is based on the unit square, and so

$$d = F_0F_0 - F_1F_{-1} = -1.$$

Also, the result in (19) becomes:

$$\begin{vmatrix} F_n & F_{n-1} \\ F_{n+1} & F_n \end{vmatrix} = (-1)^{n+1} .$$

24. Investigate all eight polygons based on unit squares at the origin. For example, in addition to polygon (23), we also have (F_{n-1}, F_n). What are the asymptotes, etc. ?

25. This material reveals a great many properties of the Fibonacci-type sequences in a very geometric and graphic fashion. One obvious and several not-so-obvious generalizations are immediately available. But these will be the subject of another article.

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