CONCAVITY PROPERTIES OF CERTAIN SEQUENCES OF NUMBERS

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A set of non-negative real numbers C_k $(k = 1, 2, \cdots, N)$ is said to be unimodal if there exists an integer n such that

$$C_k \le C_{k+1} \qquad (1 \le k < n)$$
$$C_k \ge C_{k+1} \qquad (n \le k < N) \quad .$$

A stronger property is logarithmic concavity:

$$C_k^2 \ge C_{k+1} C_{k-1} \qquad (1 < k < N) \,. \tag{1}$$

Strong logarithmic concavity (SLC) means that the inequality in (1) is strict for all k.

In a recent paper, Lieb [1] has proved that the Stirling numbers of the second kind

$$S(N,k) = \frac{1}{k!} \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} j^N$$

have the SLC property. The proof makes use of Newton's inequality. If the polynomial

$$Q(x) = \sum_{k=1}^{N} C_k x^k \tag{2}$$

has only real roots, then

$$C_k^2 \ge \frac{k(N - k + 1)}{(k - 1)(N - k)} C_{k+1} C_{k-1} \qquad (1 < k < N) \,.$$

In view of the above, it is of some interest to exhibit sequences $\{C_k\}$ with the SLC property for which the corresponding polynomial does not have the SLC property. Such an example is furnished by

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(3) $$(1 + x + x^2)^n = \sum_{k=0}^{2n} c(n,k)x^k .$$

It follows at once from (3) that $c(n,k)$ satisfies the recurrence

(4) $$c(n + 1,k) = c(n,k - 2) + c(n,k - 1) + c(n,k) .$$

We shall show first that, for $n \geq 2$,

(5) $$c(n,k) < c(n,k + 1) \qquad (0 \leq k < n) ,$$

(6) $$c(n,k) > c(n,k + 1) \qquad (n \leq k < 2n) .$$

Since

(7) $$c(n,k) = c(n,2n - k) ,$$

(5) and (6) are equivalent so that it suffices to prove (5). Since

$$(1 + x + x^2)^2 = 1 + 2x + 3x^2 + 2x^3 + x^2 ,$$

it is clear that (5) holds for $n = 2$. Assume that (5) holds for $2 \leq n \leq m$. Then, for $k < m$,

$$c(m + 1,k + 1) - c(m + 1,k) = c(m,k + 1) - c(m,k - 2) > 0 .$$

For $k = m$ we have

$$\begin{aligned} &c(m + 1,m + 1) - c(m + 1,m) \\ &\quad = 2c(m,m - 1) + c(m,m) - [c(m,m - 2) + c(m,m - 1) + c(m,m)] \\ &\quad = c(m,m - 1) - c(m,m - 2) > 0 . \end{aligned}$$

This completes the proof of (5).

We remark that $c(n,n)$ satisfies

$$\sum_{n=0}^{\infty} c(n,n)x^n = (1 - 2x - 3x^2)^{-\frac{1}{2}} .$$

For proof see [2, p. 126, No. 217].

We shall now show that, for $n \geq 2$,

(8) $$c^2(n,k) > c(n,k + 1)c(n,k - 1) \qquad (0 < k < 2n) .$$

This holds for $n = 2$. We assume that (8) holds for $2 \leq n \leq m$.

Note that (8) implies

$$(9) \qquad c(n,j)c(n,k) > c(n,j-1)c(n,k+1) \qquad (0 < j \leq k < 2n) .$$

Indeed, by (8)

$$\frac{c(n,k)}{c(n,k+1)} > \frac{c(n,k-1)}{c(n,k)} ,$$

which implies

$$\frac{c(n,k)}{c(n,k-1)} \quad \frac{c(n,j-1)}{c(n,j)} .$$

Thus, for $0 < k < 2m$,

$$\begin{vmatrix} c(m+1,k) & c(m+1,k+1) \\ c(m+1,k-1) & c(m+1,k) \end{vmatrix}$$

$$= \begin{vmatrix} c(m,k-2) + c(m,k-1) + c(m,k) & c(m,k-1) + c(m,k) + c(m,k+1) \\ c(m,k-3) + c(m,k-2) + c(m,k-1) & c(m,k-2) + c(m,k-1) + c(m,k) \end{vmatrix}$$

$$= \begin{vmatrix} c(m,k-2) & c(m,k-1) \\ c(m,k-3) & c(m,k-2) \end{vmatrix} + \begin{vmatrix} c(m,k-2) & c(m,k) \\ c(m,k-3) & c(m,k-1) \end{vmatrix}$$

$$+ \begin{vmatrix} c(m,k-2) & c(m,k+1) \\ c(m,k-3) & c(m,k) \end{vmatrix} + \begin{vmatrix} c(m,k-1) & c(m,k) \\ c(m,k-2) & c(m,k-1) \end{vmatrix}$$

$$+ \begin{vmatrix} c(m,k-1) & c(m,k+1) \\ c(m,k-2) & c(m,k) \end{vmatrix} + \begin{vmatrix} c(m,k) & c(m,k-1) \\ c(m,k-1) & c(m,k-2) \end{vmatrix}$$

$$+ \begin{vmatrix} c(m,k) & c(m,k+1) \\ c(m,k-1) & c(m,k) \end{vmatrix} .$$

The fourth and sixth determinants cancel while each of the remaining five is positive by (9). Hence

$$c^2(m+1,k) > c(m+1,k-1)c(m+1,k) \qquad (0 < k < 2m) .$$

As for the excluded values, we have by (7)

$$c^2(m+1,2m) - c(m+1,2m-1)c(m+1,2m-1) = c^2(m+1,2) - c(m+1,3)c(m+1,1) > 0,$$
$$c^2(m+1,2m+1) - c(m+1,2m)c(m+1,2m+2) = c^2(m+1,1) - c(m+1,2)c(m+1,0) > 0 .$$

In a similar way we can show that the coefficients of $c_r(n,k)$ defined by

$$(1 + x + \cdots + x^r)^n = \sum_{k=0}^{nr} c_r(n,k)\, x^k$$

have the SLC property for $n \geq 2$.

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