A CONJECTURE CONCERNING LUCAS NUMBERS

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Anaya and Crump (now Anaya and Anaya) [1] have proved that

$$[a^k F_n + \tfrac{1}{2}] = F_{n+k} \qquad (n \geq k \geq 1),$$

where $a = \frac{1}{2}(1+\sqrt{5})$ and $[x]$ denotes the greatest integer $\leq x$. They remark that it seems reasonable that

$$[a^k L_n + \tfrac{1}{2}] = L_{n+k},$$

when n is somewhat greater than k.

We shall show that

$$(1) \qquad [a^k L_n + \tfrac{1}{2}] = L_{n+k} \qquad (n \geq k+2,\ k \geq 2).$$

Moreover, for $k = 1$,

$$(2) \qquad [a L_n + \tfrac{1}{2}] = L_{n+1} \qquad (n \geq 4).$$

To prove (1), it suffices to show that

$$(3) \qquad \left| a^k L_n - L_{n+k} \right| < \tfrac{1}{2} \qquad (n \geq k+2,\ k \geq 2),$$

that is,

$$(4) \qquad \left| b^n (a^k - b^k) \right| < \tfrac{1}{2} \qquad (n \geq k+2,\ k \geq 2),$$

where we have used

$$L_n = a^n + b^n, \qquad b = \tfrac{1}{2}(1 - \sqrt{5}).$$

Clearly (4) is satisfied if

$$a^{-n}(a^k + a^{-k}) < \tfrac{1}{2} \qquad (n \geq k+2,\ k \geq 2).$$

Thus it is enough to show that

$$a^{-k-2}(a^k + a^{-k}) < \tfrac{1}{2} \qquad (k \geq 2),$$

that is,

$$(5) \qquad a^{-2} + a^{-2k-2} < \tfrac{1}{2} \qquad (k \geq 2).$$

Since

$$a^{-2} + a^{-6} = \frac{3 - \sqrt{5}}{2} - 9 - 4\sqrt{5} = \tfrac{1}{2}(21 - 9\sqrt{5}) < \tfrac{1}{2},$$

[Continued on page 550.]