ADDENDUM TO THE PAPER "FIBONACCI REPRESENTATIONS"

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1. The presentation and investigation of the functions a and b given in the paper cited in the title [1] can be simplified if we consider the following: <u>Every positive integer N has a unique representation in the form</u>

$$N = \delta_2 F_2 + \delta_3 F_3 + \cdots \, , \tag{1}$$

where δ_i is either 0 or 1 and $\delta_i \delta_{i+1} = 0$. This <u>canonical</u> or Zeckendorf representation may be written more briefly

$$N = \cdot\, \delta_2 \delta_3 \delta_4 \delta_5 \cdots \, . \tag{2}$$

Let A be the sequence of length 1 consisting of a 0, A = (0), and let B be the sequence of length 2, B = (1,0). Clearly, then, N can be written uniquely as a sequence of A's and B's, and any sequence of A's and B's, infinite on the right, containing only a finite number of B's, represents a non-negative integer. We may regard A and B as functions. For instance A(N) is to be the sequence obtained by adjoining A to the left of the sequence representing N, and similarly for B(N).

Then we see immediately that

$$N + A(N) + 1 = B(N), \qquad (N \geq 0) \, . \tag{3}$$

Now define

$$\begin{cases} a(N) = A(N - 1) + 1 & (N \geq 1) \\ b(N) = B(N - 1) + 1 & (N \geq 1) \end{cases} \, . \tag{4}$$

Then (3) becomes

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$$N + a(N) = b(N), \qquad N \geq 1. \tag{5}$$

Hence properties (2.2), (2.3) and (2.4) of [1] are easily verified, so we have, in fact,

$$\begin{cases} a(N) = [\alpha N] \\ b(N) = [\alpha^2 N], \qquad \alpha = (1 + \sqrt{5})/2 \end{cases} \tag{6}$$

as before, ((1.6) of [1]).

The advantage of introducing A and B appears when we calculate e(a) and e(b). We have

$$\begin{cases} e(a(N)) = e(A(N-1) + 1) = e(A(N-1)) + 1 = N \\ e(b(N)) = e(B(N-1) + 1) = 1 + A(N-1) = a(N) \end{cases} \tag{7}$$

The function e is defined by (1.7) in [1]:

$$e(\delta_2 F_2 + \delta_3 F_3 + \cdots) = \delta_2 F_1 + \delta_3 F_2 + \cdots. \tag{8}$$

To obtain (7) we have used the fact that e(N) is independent of the Fibonacci representation chosen for N.

It is also useful to define E(N) by means of

$$e(N) = E(N-1) + 1; \tag{9}$$

this definition may be compared with (4). Let N have the canonical representation (1) and consider

$$N + 1 = 1 + \cdot\,\delta_2\delta_3\delta_4\cdots. \tag{10}$$

If $\delta_2 = 0$ we may write

$$N + 1 = \cdot 1\,\delta_3\delta_4\cdots.$$

This representation may not be canonical. However, by (8) we have

$$e(N + 1 = 1 + \cdot\,\delta_3\delta_4\delta_5\cdots.$$

Hence, by (8) and (9),

(11) $$E(N) = \cdot\,\delta_3\delta_4\delta_5\cdots$$

If $\delta_2 = 1$, then $\delta_3 = 0$ and we get

$$N + 1 = \cdot\,01\,\delta_4\delta_5\cdots\ .$$

Again this representation may not be canonical but, by (8),

$$e(N + 1) = \cdot\,1\,\delta_4\delta_5\cdots = 1 + \cdot\,\delta_3\delta_4\delta_5\cdots\ .$$

It follows that

$$E(N) = \cdot\,\delta_3\delta_4\delta_5\cdots\ .$$

Thus in any case if N has the canonical representation (1), E(N) is determined by (11).

To sum up we state the following.

Theorem. Let N have the canonical representation

$$N = \cdot\,\delta_2\delta_3\delta_4\cdots\ .$$

Then

$$\begin{aligned} A(N) &= \cdot\,0\,\delta_2\delta_3\delta_4\cdots \\ B(N) &= \cdot\,10\,\delta_2\delta_3\delta_4\cdots \\ E(N) &= \cdot\,\delta_3\delta_4\delta_5\cdots \end{aligned}\ .$$

2. Similar observations may be made for Fibonacci representations of higher order. For instance, if we put

(12) $$A = (0), \qquad B = (10), \qquad C = (110)\ ,$$

then the relations between A, B, C and a, b, c of [2] are given by

(13) $$\begin{cases} a(N) = A(N - 1) + 1 \\ b(N) = B(N - 1) + 1 \\ c(N) = C(N - 1) + 1 \end{cases}\ ,$$

where $N \geq 1$.

3. By Theorem 11 of [1]

$$(14)\qquad \begin{cases} N \in (a) \rightleftarrows 0 < \left\{\dfrac{N}{\alpha^2}\right\} < \frac{1}{2}\,, \\[2ex] N \in (b) \rightleftarrows \dfrac{1}{\alpha} < \left\{\dfrac{N}{\alpha^2}\right\} < 1\,, \end{cases}$$

where $\{x\}$ denotes the fractional part of x. The possibility $\{N/\alpha^2\} = 1/\alpha$ never occurs.

We should like to point out that (14) can be replaced by the following slightly simpler criterion.

$$(15)\qquad \begin{cases} N \in (a) \rightleftarrows \{\alpha N\} > \dfrac{1}{\alpha^2} \\[2ex] N \in (b) \rightleftarrows \{\alpha N\} < \dfrac{1}{\alpha^2} \end{cases} .$$

As above, $\{\alpha N\} = 1/\alpha^2$ is impossible.

To see that (14) and (15) are equivalent, it suffices to observe that

$$\left\{\frac{N}{\alpha^2}\right\} = \{(2-\alpha)N\} = 1 - \{\alpha N\} .$$

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