SPECIAL CASES OF FIBONACCI PERIODICITY

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1. INTRODUCTION

This paper will deal with the periodicity of Fibonacci sequences; where the Fibonacci sequence $\{F_n\}_{n=0}^{\infty}$ is defined with $F_0 = 0$, $F_1 = 1$, and $F_{n+2} = F_{n+1} + F_n$; the Lucas sequence

$$\{L_n\}_{n=0}^{\infty}$$

is defined with $L_0 = 2$, $L_1 = 1$, and $L_{n+2} = L_{n+1} + L_n$; and the generalized Fibonacci sequence $\{H_n\}_{n=0}^{\infty}$ has any two starting values with $H_{n+2} = H_{n+1} + H_n$. We will see that in one case, that of modulo 2^n, all generalized Fibonacci sequences will have the same period. In a second case, that of modulo 5^n, different sequences will have different periods. We will also consider the periods modulo 10^n. In each case except that of 10^n, the method of proof will be to show that with sequence $\{A_n\}$, modulus m, and period p, then $A_{n+p} \equiv A_n \pmod{m}$ and $A_{n+1+p} \equiv A_{n+1} \pmod{m}$. Identities in the proof may be found in [1].

2. THE FIBONACCI CASE MOD 2^n

Theorem 1. The period of the Fibonacci sequence modulo 2^n is $3 \cdot 2^{n-1}$. We will prove that: (A) $F_{3\cdot 2^{n-1}} \equiv F_0 \pmod{2^n}$ and (B) $F_{3\cdot 2^{n-1}+1} \equiv F_1 \pmod{2^n}$.

A. The proof is by induction.

(1) When n = 1, $F_{3\cdot 2^{\ell-1}} = F_3 = 2 \equiv 0 \pmod{2^\ell}$.

(2) Suppose $F_{3\cdot 2^{k-1}} \equiv 0 \pmod{2^k}$.

(3) Now, $F_{3\cdot 2^k} = F_{3\cdot 2^{k-1}} L_{3\cdot 2^{k-1}}$
from the identity $F_{2n} = F_n L_n$.

(4) We claim $L_{3k} \equiv 0 \pmod{2}$.
The proof is by induction.

(5) When k = 1, $L_{3\cdot 1} = 4 \equiv 0 \pmod{2}$.

(6) Suppose $L_{3m} \equiv 0 \pmod{2}$.

(7) $L_{3(m+1)} = 2L_{3m+1} + L_{3m} \equiv 0 \pmod{2}$
and statement (4) is established.

Using (3), with the induction hypothesis (2), and (4), it follows that

(8) $F_{3\cdot 2^k} \equiv 0 \pmod{2^{k+1}}$
and Part A is proved.

B. (9) First, $F_{3\cdot 2^{n-1}+1} = (F_{3\cdot 2^{n-2}+1})^2 + (F_{3\cdot 2^{n-2}})^2$
using the identity $F_{m+n+1} = F_{m+1}F_{n+1} + F_mF_n$. Now, since $F_{3\cdot 2^{n-1}} \equiv 0 \pmod{2^{n-1}}$ from Part A, it follows that

(10) $(F_{3.2^{n-2}})^2 \equiv 0 \pmod{2^n}$.

(11) Also $(F_{3.2^{n-2}+1})^2 \equiv 1 \pmod{2^n}$

from the identity $F_{n+1}F_{n-1} - F_n^2 = (-1)^n$ and (10).

Part B follows from these three steps.

3. THE GENERAL FIBONACCI CASE MOD 2^n

Theorem 2. The period of any generalized Fibonacci sequence modulo 2^n is $3 \cdot 2^{n-1}$. We will prove that: (A) $H_{3.2^{n-1}+1} \equiv H_1 \pmod{2^n}$ and (B) $H_{3.2^{n-1}+2} \equiv H_2 \pmod{2^n}$.

A. We will have to consider three cases.

Case 1: $n = 1$. $H_{3.2^{1-1}+1} = H_4 = 2H_2 + H_1 \equiv H_1 \pmod{2^1}$.

Case 2: $n = 2$. $H_{3.2^{2-1}+1} = H_7 = 3H_2 + 5H_1 \equiv H_1 \pmod{2^2}$.

Case 3: $n > 2$.

(12) First, $H_{3.2^{n-1}+1} = H_{3.2^{n-2}+1}F_{3.2^{n-2}+1} + H_{3.2^{n-2}}F_{3.2^{n-2}}$,
from the identity $H_{m+n+1} = H_{m+1}F_{n+1} + H_mF_n$.

(13) We need the fact that $F_{3.2^{n-2}} \equiv 0 \pmod{2^n}$ for $n > 2$, which can be proved by induction in the manner of the proof of 1-A.

(14) Next we claim $H_{3.2^{n-2}}F_{3.2^{n-2}+1} \equiv H_1 \pmod{2^n}$ for $n > 2$.
Since $H_{n+1} = H_1F_{n-1} + H_2F_n$, we can multiply both sides by F_{n+1}

(15) so
$$H_{3.2^{n-2}+1}F_{3.2^{n-2}+1} = H_1F_{3.2^{n-2}-1}F_{3.2^{n-2}+1} + H_2F_{3.2^{n-2}}F_{3.2^{n-2}+1}.$$

(16) Now, $F_{3.2^{n-2}-1}F_{3.2^{n-2}+1} \equiv 1 \pmod{2^n}$ $n > 2$
using the identity $F_{n+1}F_{n-1} - F_n^2 = (-1)^n$ and (13).

Our claim in (14) follows from (15), (16), and (13) and Case 3 follows from (12), (13), and (16).

B. (17) First, $H_{3.2^{n-1}+2} = H_1F_{3.2^{n-1}} + H_2F_{3.2^{n-1}+1}$
from the identity $H_{n+2} = H_1F_n + F_2F_{n+1}$.
Since $F_{3.2^{n-1}} \equiv 1 \pmod{2^n}$ from 1-A, and $F_{3.2^{n-1}+1} \equiv 1 \pmod{2^n}$ from 1-B, Part B follows immediately.

One of the key parts in the proof of Theorem 1 is being able to write $F_{3.2^k}$ in terms of $F_{3.2^{k-1}}$ as in statement (3). For the next theorem, an analogous result is needed for $F_{5^{n+1}}$ in terms of F_{5^n}.

4. THE FIBONACCI CASE MOD 5^n

We need a simple lemma.

Lemma. $F_{5^{n+1}} = F_{5^n}(L_{4.5^n} - L_{2.5^n} + 1)$, $n = 1, 2, \cdots$.

Proof. We will use the Binet forms

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad \text{and} \quad L_n = \alpha^n + \beta^n ,$$

where

$$\alpha = \frac{1+\sqrt{5}}{2} \quad \text{and} \quad \beta = \frac{1-\sqrt{5}}{2} .$$

Note that $\alpha\beta = -1$.

$$F_{5^{n+1}} = \frac{\alpha^{5^{n+1}} - \beta^{5^{n+1}}}{\alpha - \beta} = \frac{\alpha^{5^n \cdot 5} - \beta^{5^n \cdot 5}}{\alpha - \beta}$$

$$= \frac{(\alpha^{5^n} - \beta^{5^n})}{\alpha - \beta}(\alpha^{5^n \cdot 4} + \alpha^{5^n \cdot 3}\beta^{5^n} + \alpha^{5^n \cdot 2} + \beta^{5^n \cdot 2} + \alpha^{5^n}\beta^{5^n \cdot 3} + \beta^{5^n \cdot 4})$$

$$= \frac{(\alpha^{5^n} - \beta^{5^n})}{\alpha - \beta}[\alpha^{5^n \cdot 4} + \beta^{5^n \cdot 4} + (\alpha\beta)^{5^n}(\alpha^{5^n \cdot 2} + \beta^{5^n \cdot 2}) + (\alpha\beta)^{5^n \cdot 2}]$$

$$= F_{5^n}(L_{5^n \cdot 4} - L_{5^n \cdot 2} + 1) .$$

Theorem 3. The period of the Fibonacci numbers modulo 5^n is $4 \cdot 5^n$.

Proof. We will prove that: (A) $F_{4 \cdot 5^n} \equiv F_0 \pmod{5^n}$ and (B) $F_{4 \cdot 5^n + 1} \equiv F_1 \pmod{5^n}$.

A. (18) Since $F_n | F_{kn}$, $\quad F_{4 \cdot 5^n} \equiv F_{5^n} \pmod{5^n}$

(19) Next we claim $\quad F_{5^n} \equiv 0 \pmod{5^n}$.

The proof is by induction.

(20) When $n = 1$, $\quad F_{5^1} \equiv F_5 = 5 \equiv 0 \pmod{5^1}$.

(21) Suppose $\quad F_{5^k} \equiv 0 \pmod{5^k}$.

(22) Now, $F_{5^{k+1}} = F_{5^k}(L_{4 \cdot 5^k} - L_{2 \cdot 5^k} + 1)$ from the Lemma.

(23) $\quad L_{4 \cdot 5^k} \equiv 2 \pmod 5$

from the identity $L_{4n} - 2 = 5F_{2n}^2$,

(24) and $\quad L_{2 \cdot 5^k} \equiv -2 \pmod 5$

from the identity $L_{2(2n+1)} + 2 = 5F_{2n+1}^2$.

Using the induction hypothesis (21), with (22), (23), and (24),

(25) $\quad F_{5^{k+1}} \equiv 0 \pmod{5^{k+1}}$

and Part A follows.

B. (26) First $\quad F_{4 \cdot 5^n + 1} = (F_{2 \cdot 5^n + 1})^2 + (F_{2 \cdot 5^n})^2$

using the identity $F_{m+n+1} = F_{m+1}F_{n+1} + F_mF_n$.

From (19) it follows that

(27) $\quad (F_{2 \cdot 5^n})^2 \equiv 0 \pmod{5^n}$.

(28) Also $\quad (F_{2 \cdot 5^n + 1})^2 \equiv 1 \pmod{5^n}$

using the identity $F_{n+1}F_{n-1} - F_n^2 = (-1)^n$ and (27).

Consequently Part B is proved.

5. THE LUCAS CASE MOD 5^n

Theorem 4. The period of the Lucas numbers modulo 5^n is $4 \cdot 5^{n-1}$.

Proof. We will prove that: (A) $L_{4 \cdot 5^{n-1}} \equiv L_0 \pmod{5^n}$ and (B) $L_{4 \cdot 5^{n-1}+1} \equiv L_1 \pmod{5^n}$.

A. (29) First $L_{4\cdot 5^{n-1}} = 5(F_{2\cdot 5^{n-1}})^2 + 2$

from the identity $L_{4n} - 2 = 5F_{2n}^2$.

From (19) it can be shown that

(30) $(F_{2\cdot 5^{n-1}})^2 \equiv 0 \pmod{5^{n-1}}$.

(31) So $5(F_{2\cdot 5^{n-1}})^2 \equiv 0 \pmod{5^n}$

and Part A is proved.

B. (32) First $L_{4.5^{n+1}+2} = 5(F_{2.5^{n-1}+1})^2 - 2$

from the identity $L_{4n+2} = 5F_{2n+1}^2 - 2$.

(33) In a method similar to that used in showing (28), it can be shown that

$$(F_{2\cdot 5^{n-1}+1})^2 \equiv 1 \pmod{5^n}.$$

(34) Therefore $L_{4.5^{n-1}+2} \equiv 3 \pmod{5^n}$.

(35) From A and (34), $L_{4\cdot 5^{n-1}+2} - L_{4\cdot 5^{n-1}} \equiv 1 \pmod{5^n}$

(36) $L_{4.5^{n-1}+1} \equiv 1 \pmod{5^n}$ since $L_{n+2} = L_{n+1} + L_n$.

As shown in [2], the periods of the Fibonacci sequences modulo 10^n will be the least common multiple of the periods mod 2^n and mod 5^n. A summary of the periods is below.

Sequence	mod 2^n $n = 1, 2, \cdots$	mod 5^n $n = 1, 2, \cdots$	mod 10	mod 100	mod 10^n $n = 3, 4, \cdots$
Fibonacci $\{F_n\}$	$3\cdot 2^{n-1}$	$4\cdot 5^n$	60	300	$15\cdot 10^{n-1}$
Lucas $\{L_n\}$	$3\cdot 2^{n-1}$	$4\cdot 5^{n-1}$	12	60	$3\cdot 10^{n-1}$
Generalized Fibonacci $\{H_n\}$	$3\cdot 2^{n-1}$	variable	variable	variable	variable

6. SOME PARTING OBSERVATIONS

We note in passing that we have found some solutions to $n \mid F_n$ in the statement $F_{5^n} \equiv 0 \bmod 5^n$. To this we add two statements also involving solutions to $L_n \equiv 0 \bmod n$.

Theorem: $L_n \equiv 1 \bmod n$ for n a prime.

Theorem: $L_{2\cdot 3^k} \equiv 0 \bmod 2\cdot 3^k$, $k = 1, 2, 3, \cdots$

Theorem: $F_{2^2\cdot 3^k} \equiv 0 \bmod 2^2\cdot 3^k$, $k = 1, 2, 3, \cdots$

A new paper by Hoggatt and Bicknell will further discuss these ideas.

REFERENCES

1. V. E. Hoggatt, Jr., Fibonacci and Lucas Numbers, Houghton Mifflin, Boston, 1969.
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3. V. E. Hoggatt, Jr., "A Type of Periodicity for Fibonacci Numbers," Math. Magazine, Jan.-Feb. 1955, p. 139.
4. Frederick Gebhardt, "Generating Pseudo-Random Numbers by Shuffling a Fibonacci Sequence," Mathematics of Computation, Oct. 1967, pp. 708-709.

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