A PRIMER FOR THE FIBONACCI NUMBERS
PART X: ON THE REPRESENTATION OF INTEGERS

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The representation of integers is a topic that has been implicit in our mathematics education from our earliest years due to the fact that we employ a positional system of notation. A number such as 35864 in base ten assumes the existence of a sequence 1, 10, 100, 1000, 10000, $\cdots$, running from right to left. The digits multiplied by the members of the sequence taken in order give the indicated integer. In this case, the representation means

$$3 \cdot 10000 + 5 \cdot 1000 + 8 \cdot 100 + 6 \cdot 10 + 4 .$$

Another way of thinking of these multipliers is this: they are the number of times various members of the sequence are being used.

It is instructive to see that such a sequence used as a base for representing integers arises naturally. Suppose we allow multipliers 0, 1, or 2. We wish to have a sequence that will enable us to represent all the positive integers and furthermore we want this sequence with the multipliers to do this uniquely; that is, for each integer there is one and only one representation by means of the sequence and the multipliers. Clearly, the first member of the sequence will have to be 1; otherwise, we could never represent the first integer 1. With this, we can represent 0, 1, or 2. Hence, the next integer we need is 3. The following table shows how at each step we are able to represent additional integers and likewise what is the next integer that is needed.

Sequence	Representations added
1	0, 1, 2
3	3, 4, 5, 6, 7, 8
9	9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, $\cdots$, 26
27	27, 28, 29, $\cdots$, 79, 80
81	81, 82, 83, $\cdots$, 241, 242

Note that, as far as we have gone, the representation is unique. Assume that we have unique representation when the sequence goes to 3^n and that this representation extends to $3^{n+1} - 1$. Adding 3^{n+1} to the sequence enables us to go from 3^{n+1} to $2 \cdot 3^{n+1} - 1$ in a unique manner, but this sum is $3^{n+2} - 1$. Thus, the base three representation of integers using the sequence 1, 3, 9, 27, 81, $\cdots$ arises naturally in the case of allowed multipliers 0, 1, 2, and the requirements of complete and unique representation.

Perhaps the most interesting case of representation is that in which the allowed multipliers are 0, 1. We build up the sequence that goes with these multipliers giving complete and unique representation.

Sequence	Representations added
1	1
2	2, 3
4	4, 5, 6, 7
8	8, 9, 10, 11, 12, 13, 14, 15
16	16, 17, 18, ⋯, 30, 31
32	32, 33, 34, ⋯, 62, 63

Thus far the representation is unique. If we have unique and complete representation when the largest term of the sequence is 2^n and the representation extends to $2^{n+1} - 1$, then on adding 2^{n+1} to the sequence, we extend complete and unique representation to $2^{n+1} + 2^{n-1} - 1 = 2^{n+2} - 1$.

Another way of thinking of representation when the multipliers are 0 and 1 is this: We have a sequence where integers are represented by distinct members of the sequence. Thus the base two integer 110111010 says that the number in question is the sum of 2^8, 2^7, 2^5, 2^4, 2^3, and 2. The powers of two along with 1 enable us to represent all integers uniquely by combining different powers of two.

INCOMPLETE AND NON-UNIQUE SEQUENCES

Let us return to the representation with multipliers 0, 1, and 2. Clearly, if instead of taking 1, 3, 9, 27, 81, ⋯, we take some larger numbers such as 1, 3, 10, 28, 82, 244, ⋯, it will not be possible to represent all integers.

Sequence	Representations added
1	1, 2
3	3, 4, 5, 6, 7, 8
10	10-18, 20-28
28	28-36, 38-46, 48-56, 56-64, 66-74, 76-84
82	82-90, 92-100, etc.

Below 100, the numbers that cannot be represented are 9, 19, 37, 47, 65, 75, and 91. On the other hand, 28, 56, 82, 83, and 84 have two representations.

Suppose that instead of making the numbers of the sequence slightly larger we make them a bit smaller. Let us take the sequence 1, 3, 8, 26, 80, 242, ⋯, as before:

Sequence	Representations added
1	1, 2
3	3, 4, 5, 6, 7, 8
8	8-16, 16-24
26	26-34, 34-42, 42-50, 52-60, 60-68, 68-76
80	80-88, 88-96, 96-104, 106-114, 114-122, 122-130,
	132-140, 140-148, 148-156, 160, etc.

Up to 160, the missing integers are 25, 51, 77, 78, 79, 105, 131, 157, 158, and 159. Duplicated integers are 8, 16, 34, 42, 60, 68, 88, 96, 114, 122, 140, and 148.

The sequence 1, 3, 8, 23, 68, 203, $\cdots$, gives complete but not unique representation.

Sequence	Representations added
1	1, 2
3	3-8
8	8-16, 16-24
23	23-31, 31-39, 39-47, 46-54, 54-62, 62-70
68	68-76, 76-84, 84-92, 91-99, 99-107, 107-115,
	114-122, 122-130, 130-138, 136-144, 144-152, etc.

Up to 140 there is complete representation but duplicate representation for the following: 8, 16, 23, 24, 31, 39, 46, 47, 54, 62, 68, 69, 70, 76, 84, 91, 92, 99, 107, 114, 115, 122, 130, 136, 137, and 138.

FIBONACCI REPRESENTATIONS

Let us now consider the case in which the multipliers are 0, 1 and the basic sequence is the Fibonacci sequence 1, 1, 2, 3, 5, 8, 13, $\cdots$. That this sequence gives complete representation is not difficult to prove. In fact, the representation is still complete if we eliminate the first 1 and use the sequence 1, 2, 3, 5, 8, 13, $\cdots$. In the table following, note that the representation at each stage gives complete representation up to and including $F_{n+2} - 2$. Assume this to be so up to a certain F_n. Then upon adjoining F_{n+1} to the sequence the representation will be complete to $F_{n+1} + F_{n+2} - 2$, which is much beyond F_{n+2}, the next term to be added. Thus the representation is complete, but it is evidently not unique.

Sequence	Representations added
1	1
2	2, 3
3	3, 4, 5, 6
5	5-8, 8-11
8	8-11, 11-14, 13-16, 16-19
13	13-16, 16-19, 18-21, 21-24, 21-24, 24-27, 26-29, 29-32

AN INTERESTING THEOREM

To get a new perspective on representation by this Fibonacci sequence we write down the representations of the integers in their various possible forms. (Read 10110 as 8 + 3 + 2 or $1 \cdot F_6 + 0 \cdot F_5 + 1 \cdot F_4 + 1 \cdot F_3 + 0 \cdot F_2$.)

Integer	Representations	Integer	Representations
1	1	11	10100, 10011, 1111
2	10	12	10101
3	11, 100	13	11000, 10110, 100000
4	101	14	100001, 11001, 10111
5	110, 1000	15	100010, 11010
6	111, 1001	16	100100, 100011, 11100, 11011
7	1010	17	100101, 11101
8	1100, 1011, 10000	18	101000, 100110, 11110
9	10001, 1101	19	101001, 100111, 11111
10	10010, 1110	20	101010

Now the Fibonacci sequence has the property that the sum of two consecutive members of the sequence gives the next member of the sequence. Accordingly, one might argue, it is superfluous to have two successive members of the sequence in a representation since they can be combined to give the next member. If this is done, we arrive at representations in which there are no two consecutive ones in the representation. Looking over the list of integers that we have represented thus far, it appears that there is just one such representation for each integer in this form.

Suppose we go at this from another direction. We are building up a sequence that will represent the integers uniquely with multipliers 0 and 1. However, we stipulate that no two consecutive members of the sequence may be found in any representation. We form a table as before.

Sequence	Representations added
1	1
2	2
3	3, 4
5	5, 6, 7
8	8, 9, 10, 11, 12
13	13, 14, 15, 16, 17, 18, 19, 20

To this point the representation is unique and the sequence that is emerging is the Fibonacci sequence 1, 2, 3, 5, 8, 13, $\cdots$. Assume that up to F_n there is unique representation to $F_{n+1} - 1$. On adding F_{n+1} to the sequence, we cannot use F_n in conjunction with it but only terms up to F_{n-1}. But by supposition these may represent all integers up to $F_n - 1$ in a unique way. Hence with F_{n+1} we can represent uniquely all integers from F_{n+1} to $F_{n+1} + F_n - 1 = F_{n+2} - 1$. Hence the uniqueness and completeness of this type of representation are established, which is known as Zeckendorf's Theorem.

MORE ZEROES IN THE REPRESENTATION

A natural question to ask is: Would it be possible to require that there be at least two zeroes between 1's in the representation and obtain unique representation? We can build up the sequence as before taking into account this requirement.

Sequence	Representations added
1	1
2	2
3	3
4	4, 5
6	6, 7, 8
9	9, 10, 11, 12
13	13, 14, 15, 16, 17, 18
19	19, 20, 21, 22, 23, 24, 25, 26, 27
28	28-40

Up to this point, the representation is complete and unique. We have a sequence, but it would be difficult to operate with it unless we knew the way it builds up according to some recursion relation. The relation appears as

$$T_{n+1} = T_n + T_{n-2} .$$

Now assume that up to T_n we have unique representation to $T_{n+1} - 1$, where T_{n+1} is given by the recursion relation in terms of previous members of the sequence. Then on adding T_{n+1} to the sequence we may not use T_n or T_{n-1} in conjunction with it but only terms up to T_{n-2}. But these give unique and complete representation to $T_{n-1} - 1$. Hence upon adding T_{n+1} to the sequence we have extended unique and complete representation from T_{n+1} to $T_{n+1} + T_{n-1} - 1 = T_{n+2} - 1$. Thus, the uniqueness and completeness are established in general.

The sequences required for unique and complete representation when three, four, or more zeroes are required between 1's in the representation can be built up in the same way. Some are listed on the following page.

Zeroes	Sequence derived	Recursion relation
3	1, 2, 3, 4, 5, 7, 10, 14, 19, 26, 36, 50, 69, 95, 131, 181, 250, ···	$T_{n+1} = T_n + T_{n-3}$
4	1, 2, 3, 4, 5, 6, 8, 11, 15, 20, 26, 34, 45, 60, 80, 106, 140, 185, ···	$T_{n+1} = T_n + T_{n-4}$
5	1, 2, 3, 4, 5, 6, 7, 9, 12, 16, 21, 27, 34, 43, 55, 71, 92, 119, ···	$T_{n+1} = T_n + T_{n-5}$
6	1, 2, 3, 4, 5, 6, 7, 8, 10, 13, 17, 22, 28, 35, 43, 53, 66, 83, 105, 133, ···	$T_{n+1} = T_n + T_{n-6}$

For k zeroes, the sequence is $1, 2, 3, 4, \cdots, k, k+1, k+2,$ which enables us to get $k+3$; then $k+4$ which gives $k+5$, $k+6$; and so on. Up to this point the representation is unique and complete; the recursion relation beginning with $k+2$ is $T_{n+1} = T_n + T_{n-k}$. Assume that the sequence up to T_n gives unique and complete representation to $T_{n+1} - 1$. Then upon adding T_{n+1} the highest term we can use in conjunction with it is $T_{n+1-k-1} = T_{n-k}$ which gives unique representation to $T_{n-k+1} - 1$ by hypothesis. Hence upon adding T_{n+1} we have unique representation from T_{n+1} to $T_{n+1} + T_{n-k+1} - 1 = T_{n+2} - 1$.

MULTIPLIERS 0, 1, 2

We know that we obtain unique and complete representation using multipliers 0, 1, 2 when we have the geometric progression $1, 3, 9, 27, \cdots$. Can we find a unique and complete representation if we demand that there be a zero between any two non-zero digits in the representation? Let us build this up as before.

Sequence	Representations added
1	1, 2
3	3, 6
4	4, 5, 6, 8, 9, 10
7	7, 8, 9, 10, 13, 14, 15, 16, 17, 20
11	11-14, 17, 15-17, 19-25, 28, 26-28, 30-32
18	18-21, 24, 22-24, 26-28, 25-28, 31-35, 38, 36-39, 42, 40-42, 44-46, 43-46, 49-53, 56

It appears that the sequence is the Lucas numbers. The representation is not unique. But a Lucas number L_n allows complete representation to the next Lucas number L_{n+1} (and beyond) without any additional Lucas numbers being represented. Assume that this is the case up to a certain n. Upon adding L_{n+1} we may not use L_n. Going back to L_{n-1} and preceding terms we can represent all integers up to $L_n - 1$ without being able to represent any Lucas numbers $L_n, L_{n+1}, \cdots$. Thus adding L_{n+1} allows the representation of numbers L_{n+1} to $L_{n+1} + L_n - 1 = L_{n+2} - 1$, but does not give L_{n+2} since this would require L_n. If we use $2L_{n+1}$ we would need L_n to get L_{n+3}, but since we do not have L_n it is not possible to arrive at this Lucas number. To dispose of L_{n+4} and higher Lucas numbers, we have to set a bound on the highest number at which we may arrive. Starting with L_{n-1} and working backward, the highest sum we can have is twice the sum of alternate terms beginning with L_{n-1}. If $n - 1$ is odd, this sum is $2(L_n - 2)$, and if $n - 1$ is even, this sum is $2(L_n - 1)$. In either case, the sum is less than $2L_n$. Hence an upper bound for terms when L_{n+1} is added to the sequence is $2L_{n+1} + 2L_n = 2L_{n+2}$. But $L_{n+4} = 2L_{n+2} + L_{n+1}$ which is greater than $2L_{n+2}$. Hence it is not possible to arrive at L_{n+4} or higher Lucas numbers.

This result was very encouraging and led to an investigation of cases with multipliers 0, 1, 2, 3; then 0, 1, 2, 3, 4; etc., where we still require one zero between non-zero digits. The first few terms looked interesting.

Multipliers 0, 1, 2, 3:	1, 4, 5, 9, 14, ···
Multipliers 0, 1, 2, 3, 4:	1, 5, 6, 11, 17, ···
Multipliers 0, 1, 2, 3, 4, 5:	1, 6, 7, 13, 20, ···

Unfortunately, in the sequence 1, 4, 5, 9, 14, ···, if we continue with the terms 23, 27, 60, we find that 60 is already represented by 14 and lower terms. In the sequence 1, 5, 6, 11, 17, 28, ···, the 28 is represented by earlier terms. We have run into a DRY HOLE.

Next, keeping the multipliers 0, 1, 2, the case of two zeroes between non-zero digits was investigated. This led to the sequence 1, 3, 4, 5, 9, 13, 22, 31, 53, 75, 128, 181, ···, where there are two apparent laws of formation, one for odd-numbered terms, and a second for even-numbered terms,

$$T_{2n+1} = T_{2n} + T_{2n-1}, \tag{1}$$

$$T_{2n+2} = T_{2n+1} + T_{2n-1}. \tag{2}$$

There are equivalent representations of these relations. By (1) and (2),

$$T_{2n+1} = (T_{2n-1} + T_{2n-3}) + T_{2n-1} = 2T_{2n-1} + T_{2n-3}, \tag{3}$$

$$T_{2n+2} = (T_{2n} + T_{2n-1}) + T_{2n-1} = T_{2n} + 2T_{2n-1}. \tag{4}$$

Since by (1) $T_{2n-1} = T_{2n+1} - T_{2n}$, we have from (4) $T_{2n+2} = 2T_{2n+1} - T_{2n}$, or,

$$2T_{2n+1} = T_{2n+2} + T_{2n}. \tag{5}$$

Therefore, by using (5) to express $2T_{2n-1}$ in (4),

$$T_{2n+2} = 2T_{2n} + T_{2n-2}. \tag{6}$$

Hence, combining (3) and (6), there is one recursion relation for the entire sequence,

$$T_{n+1} = 2T_{n-1} + T_{n-3}. \tag{7}$$

The manner in which the sequence builds up is shown by the following table.

Sequence	Representations added
1	1, 2
3	3, 6
4	4, 8
5	5, 6, 7, 10, 11, 12
9	9-12, 15, 18-21, 24
13	13-16, 19, 17, 21, 26-29, 32, 30, 34
22	22-25, 28, 26, 30, 27-29, 32-34, 44-47, 50, 48, 52, 49-51, 54-56

To show that the sequence will continue to be built up in this way we note the following as a basis for our induction:

(1) Adding a term T_k covers all representations up to $T_{k+1} - 1$.

(2) Adding another term of the sequence does not give additional terms of the representing sequence.

(3) The largest term that can be represented by adding T_k is less than T_{k+3}.

Now, if the above is true to T_n, add the term T_{n+1}. We can use only terms to T_{n-2} and smaller in the sequence in conjunction with T_{n+1}. Such terms can represent values up to $T_{n-1} - 1$. Hence adding T_{n+1} enables us to represent values from T_{n+1} to $T_{n+1} + T_{n-1} - 1$, which gives $T_{n+2} - 1$ if $n + 1$ is odd. If $n + 1$ is even, $T_{n+1} + T_{n-1} - 1 = 2T_{n+2} - 1$. Hence all representations up to $T_{n+2} - 1$ are covered.

On adding T_{n+1} to the sequence we do not obtain any other sequence terms. For $T_{n+2} = T_{n+1} + T_{n-1}$ and $T_{n+2} = 2T_{n+1} + T_{n-1}$ if $n + 1$ is odd, and T_{n-1} is not available in conjunction with T_{n+1}. Similarly, if $n + 1$ is even, $T_{n+2} = T_{n+1} + T_n$ and $T_{n+3} = 2T_{n+1} + T_{n-1}$ where neither T_n nor T_{n-1} is available. Finally, T_{n+4} is larger than any term that can be formed using T_{n+1} and smaller terms.

CONCLUSION

A great deal of work has been done on representations of integers in recent years. Much of this has appeared in the Fibonacci Quarterly which has published some two dozen articles totalling approximately 300 pages by such mathematicians as Carlitz, Brown, Hoggatt, Ferns, Klarner, Daykin, and others. The number of byways that may be investigated is great. It could be the project of a lifetime.