TRIANGULAR ARRAYS SUBJECT TO MAC MAHON'S CONDITIONS

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1. INTRODUCTION

We consider triangular arrays (n_{ij}) $(j = i(1)k,\ i = 1(1)k)$ and (a_{rs}) $(s = 1(1)k + 1 - r,\ r = 1(1)k)$ and let $T(n,k)$ and $C(n,k)$, respectively, denote the number of these arrays in which the entries are non-negative integers subject to the conditions

$$n_{ij} \geq n_{i,j+1}, \qquad n_{ij} \geq n_{i+1,j}, \qquad n_{11} \leq n \tag{1.1}$$

$$a_{rs} \geq a_{r,s+1}, \qquad a_{rs} \geq a_{r+1,s}, \qquad a_{11} \leq n\ . \tag{1.2}$$

The conditions (1.1) and (1.2) are the same as MacMahon [3] imposed on multi-rowed partitions. Rectangular arrays subject to these conditions have been considered by Carlitz and Riordan [1].

It is easy to evaluate $T(1,k)$ and $C(1,k)$. Indeed, taking row sums, we find that $T(1,k)$ is the number of sequences $j_1, \cdots, j_k$ with $j_i > j_{i+1}$ and $j_1 \leq k$. It follows that $T(1,k) = 2^k$. In the same way, we find that $C(1,k)$ is the number of sequences $j_1, \cdots, j_k$ with $k + 1 - i \geq j_i \geq j_{i+1}$. Hence $C(1,k)$ is the familiar Catalan number (c. f. [2])

$$C(1,k) = \frac{1}{k+2}\binom{2k+2}{k+1}\ . \tag{1.3}$$

It will be convenient to have an alternative description of $C(n,k)$ and $T(n,k)$. With each array counted by $T(n,k)$ we associate the $n \times k$ array $M = (m_{ij})$, where m_{ij} is the number of elements in the j^{th} row which are greater than or equal to i. Similarly, with each array counted by $C(n,k)$, associate the $n \times k$ array $B = (b_{ij})$, where b_{ij} is the number of elements in the j^{th} column which are greater than or equal to i. That is, $m_{ij} = \text{card}\{n_{jt} | n_{jt} \geq i\}$ and $b_{ij} = \text{card}\{a_{tj} | a_{tj} \geq i\}$. It then follows that the entries of the associated array are subject to the conditions

$$m_{ij} \geq m_{i,j+1}, \qquad m_{ij} \geq m_{i+1,j}, \qquad m_{11} \leq k\ , \tag{1.4}$$

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(1.5) $$b_{ij} \geq b_{i,j+1}, \qquad b_{ij} \geq b_{i+1,j}, \qquad b_{ij} \leq k + 1 - j .$$

It also is not difficult to verify that the $n \times k$ arrays subject to (1.4) and (1.5) are equinumerous with those counted by $T(n,k)$ and $C(n,k)$.

Here we prove that

(1.6) $$T(2,k) = \binom{2k+1}{k} ,$$

(1.7) $$T(3,k) = 2^k \binom{2k+2}{k+1} - 2^k \binom{2k+2}{k}$$

as well as

(1.8) $$C(n,k) = \det \left[\binom{n+k+1-r}{n+r-s} \right] \qquad (r,s = 1, \cdots, k) .$$

It is also shown that

$$\sum_{n=0}^{\infty} C(n,k)x^n = A_k(x) \cdot (1-x)^{\frac{-k(k+1)}{2} - 1}$$

where $A_k(x)$ is a polynomial of degree $\frac{1}{2}k(k-1)$ with integral coefficients and which satisfies the symmetry condition

(1.9) $$x^{\frac{1}{2}k(k-1)} A_k\left(\frac{1}{x}\right) = A_k(x) .$$

2. TRIANGULAR ARRAYS

We consider triangular arrays

(2.1) $$\begin{matrix} n_{11} n_{12} \cdots n_{1k} \\ n_{22} \cdots n_{2k} \\ \cdots \\ n_{kk} \end{matrix}$$

and let $T^*(n,k)$ denote the number of these arrays with non-negative integral coefficients satisfying

(2.2) $$n_{11} = n, \qquad n_{ij} \geq n_{i,j+1}, \qquad n_{ij} \geq n_{i+1,j} ,$$

We also put

$$T(n,k) = \sum_{j=0}^{n} T^*(j,k) .$$

It is immediate that $T(0,k) = T^*(0,k) = 1$ and as observed in Section 1, it is easy to see that $T^*(1,k) = 2^k - 1$. This can also be seen by classifying the arrays according as $n_{11} = 0$ or 1 and noting that this implies the recurrence

$$T^*(1,k) = T^*(1,k - 1) + T(1,k - 1) .$$

A simple verification of the boundary conditions is then all that is necessary to anchor the induction.

Next let $Q(m_{11}, m_{21}, \cdots, m_{n1})$ denote the number of $n \times k$ arrays $M = (m_{ij})$, where the m_{ij} are subject to the conditions (1.4). It is clear from the remarks of Section 1 that

$$T^*(n,k) = \sum Q(s_1, \cdots, s_n) ,$$

where the summation extends over all n-tuples $(s_1, \cdots, s_n)$ for which $k \geq s_1 \geq \cdots \geq s_n \geq 1$. A more useful reformulation of these remarks is the observation that

$$T(n,k) = Q(k + 1, k + 1, \cdots, k + 1) . \tag{2.3}$$

For the case $n = 2$, we find that

$$Q(m,r) = 1 + \sum_{s=1}^{m-1} Q(s) + \sum_{t=1}^{r-1} \sum_{s=t}^{m-1} Q(s,t)$$

$$= 2^{m-1} + \sum_{t=1}^{r-1} \sum_{s=t}^{m-1} Q(s,t) ,$$

where we have used (2.3) for the case $n = 1$. A more convenient form of this last equation is

$$Q(m,r + 1) = \sum_{s=r}^{m} Q(s,r) .$$

It is now a simple induction to show that

$$Q(m, r + 1) = 2^{m+r-1} - \sum (2^{r-j} - 1)\binom{m + j - 1}{j} \qquad (m \geq r + 1)$$

which should be compared with [1, Eq. (1.9)]. In particular, we have

$$Q(m + 1, m + 1) = 2^{2m} - \sum_{j=0}^{m} (2^{m-j} - 1)\binom{m + j}{j}$$

$$= \binom{2m + 1}{m} .$$

It now follows from (2.3) that

$$T^*(2,k) = \binom{2k + 1}{k} . \qquad (2.4)$$

3. THE CASE n = 3

The evaluation of $T(3,k)$ is more complicated but leads to a simple result. Let $Q_c(m_{11}, m_{21}, m_{31})$ denote the number of $3 \times c$ arrays (m_{ij}) whose entries are non-increasing down each column and whose positive entries are strictly decreasing along each row. Then, according to the remarks of Section 1, we have

$$T(3,k) = Q_{k+2}(k + 1, k + 1, k + 1) . \qquad (3.1)$$

It is not difficult to show (by induction on c) that

$$Q_{c+1}(r, s, t) = \sum_{i \leq j \leq k} D_{c-2i, c-2j-1, c-2k-2} , \qquad (3.2)$$

where we put

$$D_{i,j,k} = \begin{vmatrix} \binom{r}{i} & \binom{s}{i + 1} & \binom{t}{i + 2} \\ \binom{r}{j} & \binom{s}{j + 1} & \binom{t}{j + 2} \\ \binom{r}{k} & \binom{r}{k + 1} & \binom{t}{k + 2} \end{vmatrix} .$$

In particular, for $c = m = r = s = t$, it follows from (3.1) that

$$T(3,m-1) = \sum_{i\le j\le k} \begin{vmatrix} \binom{m}{2i} & \binom{m}{2i-1} & \binom{m}{2i-2} \\ \binom{m}{2j+1} & \binom{m}{2j} & \binom{m}{2j+1} \\ \binom{m}{2k+2} & \binom{m}{2k+1} & \binom{m}{2k} \end{vmatrix}$$

$$= \sum_j \begin{vmatrix} \sum_{i\le j}\binom{m}{2i} & \sum_{i\le j}\binom{m}{2i-1} & \sum_{i\le j}\binom{m}{2i-2} \\ \binom{m}{2j+1} & \binom{m}{2j} & \binom{m}{2j+1} \\ \sum_{k\ge j}\binom{m}{2k+2} & \sum_{k\ge j}\binom{m}{2k+1} & \sum_{k\ge j}\binom{m}{2k} \end{vmatrix}$$

$$= \sum_j \begin{vmatrix} \sum_{i\le j}\binom{m}{2i} & \sum_{i\le j}\binom{m}{2i-1} & \sum_{i\le j}\binom{m}{2i-2} \\ \binom{m}{2j+1} & \binom{m}{2j} & \binom{m}{2j+1} \\ \sum_{k}\binom{m}{2k} & \sum_{k}\binom{m}{2k+1} & \sum_{k}\binom{m}{2k} \end{vmatrix}$$

$$= 2^{m-1} \sum_{i\le j} \begin{vmatrix} \binom{m}{2i} & \binom{m}{2i-1} & \binom{m}{2i-2} \\ \binom{m}{2j+1} & \binom{m}{2j} & \binom{m}{2j+1} \\ 1 & 1 & 1 \end{vmatrix}$$

$$= 2^{m-1} \sum_{i<j} \left\{ \binom{m}{2i}\binom{m}{2j} + \binom{m}{2j+1}\binom{m}{2i-2} + \binom{m}{2i-1}\binom{m}{2j-1} \right.$$

$$\left. - \binom{m}{2i-1}\binom{m}{2j+1} - \binom{m}{2i}\binom{m}{2j-1} - \binom{m}{2j}\binom{m}{2i-2} \right\}$$

$$= 2^{m-1} \left\{ \sum_j \binom{m}{j}^2 - \sum_j \binom{m}{j}\binom{m}{j+1} \right\} .$$

This reduces to

(3.3) $$T(3,k) = 2^k\binom{2k+2}{k+1} - 2^k\binom{2k+2}{k},$$

(3.4) $$T^*(3,k) = 2^k\binom{2k+2}{k+1} - 2^k\binom{2k+2}{k} - \binom{2k+1}{k}.$$

It appears unlikely that this method would lead to a simple result for $T(n,k)$ even though (3.2) can be generalized in an obvious manner.

4. CATALAN DETERMINANTS

We consider triangular arrays

(4.1) $$\begin{matrix} a_{11} \cdots a_{1,k-1} & a_{1k} \\ a_{21} \cdots a_{2,k-1} & \\ a_{k1} & \end{matrix}$$

and let $C(n,k)$ denote the number of these arrays with

(4.2) $$a_{11} \le n, \qquad a_{ij} \ge a_{i,j+1}, \qquad a_{ij} \ge a_{i+1,j}.$$

Then, as observed in Section 1, we have that $C(n,k)$ is also the number of $n \times k$ arrays $B = (b_{ij})$ subject to the conditions (1.5). Also, if we put $C(j_1, \cdots, j_k)$ equal to the number of arrays (4.1) with $a_{1s} = j_s$, then we find that

(4.3) $$C(j_i, \cdots, j_k) = \sum_{r_{k-1}} \cdots \sum_{r_1} C(r_1, \cdots, r_{k-1}),$$

where the i^{th} summand extends over the range $r_{k+1-i} \le r_{k-i} \le j_{k-i}$ and, for convenience, we put $r_k = 0$.

It is an easy induction to show that (4.3) is the same as

$$C(j_1, \cdots, j_k) = \det\left[\binom{j_s + k - r}{k + s - 2r}\right] \qquad (r,s = 1,2,\cdots,k-1).$$

In particular, we find that

(4.4) $$C(n,k) = \det\left[\binom{n+k+1-r}{n+r-s}\right] \qquad (r,s = 1,\cdots,k).$$

Notice that the special case (1.3) follows from (4.4) and the identity

$$\frac{1}{k+2}\binom{2k+2}{k+1} = \sum_{j=0}^{k} (-1)^j \binom{k+1-j}{j+1} \frac{1}{k+1-j}\binom{2k-2j}{k-j} .$$

In the next place if we write (4.4) in the form

$$C(n,k) = \det\left[\binom{n+k+1-r}{k+1-2r+s}\right], \tag{4.5}$$

then we can use this determinant to define $C(n,k)$ for all real numbers n. According to this definition, we find that $C(n,k)$ is a polynomial of degree $\frac{1}{2}k(k+1)$ in n and satisfies the equation

$$C(n,k) = (-1)^{\frac{1}{2}k(k+1)} C(-k-n-1,k) . \tag{4.6}$$

Hence if we put

$$C(n,k) = \sum_{j=k}^{\frac{1}{2}k(k+1)} a_{kj}\binom{n+j}{j} , \tag{4.7}$$

then we have

$$C(-k-n-1,k) = \sum_{j=k}^{\frac{1}{2}k(k+1)} a_{kj}\binom{-k-n+j-1}{j}$$

$$= \sum_{j=k}^{\frac{1}{2}k(k+1)} (-1)^j a_{kj}\binom{k+n}{j} .$$

In order to summarize these results in terms of generating functions, we first put $C_k(x) = \sum C(n,k)x^n$ and note that

$$C_k(x) = \sum_{j=k}^{\frac{1}{2}k(k+1)} a_{kj}(1-x)^{-j-1}$$

and

$$(-1)^{\frac{1}{2}k(k+1)} C_k(x) = \sum_{n=0}^{\infty} C(-k-n-1,k)x^n$$

$$= \sum_{j=k}^{\frac{1}{2}k(k+1)} (-1)^j a_{kj}\, x^{j-k} (1-x)^{-j-1} .$$

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