Send all communications regarding Elementary Problems and Solutions to Professor A. P. Hillman, Dept. of Mathematics and Statistics, University of New Mexico, Albuquerque, New Mexico 87106. Each problem or solution should be submitted in legible form, preferably typed in double spacing, on a separate sheet or sheets, in the format used below. Solutions should be received within four months of the publication date.

Contributors (in the United States) who desire acknowledgement of receipt of their contributions are asked to enclose self-addressed stamped postcards.

NOTATION: $F_0 = 0$, $F_1 = 1$, $F_{n+2} = F_{n+1} + F_n$; $L_3 = 2$, $L_4 = 1$, $L_{n+2} = L_{n+1} + L_n$.

PROBLEMS PROPOSED IN THIS ISSUE

B-244 Proposed by J. L. Hunsucker, University of Georgia, Athens, Georgia.

Let $Q$ be the $2 \times 2$ matrix

\[
\begin{pmatrix}
1 & 1 \\
1 & 0
\end{pmatrix}
\]

and let

\[
M = \begin{pmatrix}
a & b \\
c & d
\end{pmatrix}
\]

be the sum of a finite number of matrices chosen from the sequence $Q$, $Q^2$, $Q^3$, \ldots. Prove that $b = c$ and $a = b + d$.

B-245 Proposed by Richard M. Grassl, University of New Mexico, Albuquerque, New Mexico.

Show that each term $F_n$ with $n > 0$ in the sequence $F_0$, $F_1$, $F_2$, \ldots is expressible as $x^2 + y^2$ or $x^2 - y^2$ with $x$ and $y$ terms of the sequence with distinct subscripts.

B-246 Proposed by L. Carlitz, Duke University, Durham, North Carolina.

Show that at least one of the following sums is irrational:

\[
\sum_{n=0}^{\infty} \frac{1}{F_{2n+1}}, \quad \sum_{n=0}^{\infty} \frac{(-1)^n}{L_{2n+1}}.
\]
B-247 Proposed by Larry Lang, Student, San Jose State University, San Jose, California.

Given that \( m \) and \( n \) are integers with \( 0 < n < m \) and \( F_n \) \( \mid \) \( L_m \), prove that \( n \) is 1, 2, 3, or 4.

B-248 Proposed by V. E. Hoggatt, Jr., San Jose State University, San Jose, California.

Let \( k \) be a positive integer and let \( h = 5^k \). Prove that \( h \) \( \mid \) \( F_h \).

B-249 Proposed by V. E. Hoggatt, Jr., San Jose State University, San Jose, California.

Let \( k \) be a positive integer and let \( g = 2 \cdot 3^k \). Prove that \( g \) \( \mid \) \( L_g \).

SOLVERS INADVERTENTLY OMITTED FROM PREVIOUS ISSUES

J. L. Brown, Jr: B-219
Herta T. Freitag: B-202, B-203, B-206, B-207.
Graham Lord: B-202, B-203, B-204, B-205

SOLUTIONS

TWIN PRIMES SLIGHTLY DISGUISED


Let \( p_m \) be the \( m \)th prime. Prove that \( p_m \) and \( p_{m+1} \) are twin primes (i.e., \( p_{m+1} = p_m + 2 \)) if and only if

\[
\sum_{n=1}^{m} (p_{n+1} - p_n) = p_m.
\]


The sum telescopes to \( p_{m+1} - p_1 = p_{m+1} - 2 \).


SIMPLE SUBSTITUTION IN A CONVERGENT SERIES

B-221 Proposed by R. Garfield, College of Insurance, New York, New York.

Prove that

\[
\sum_{n=1}^{\infty} \frac{1}{F_n L_n} = \sum_{n=1}^{\infty} \frac{1}{F_{2n}}.
\]
Solution by Wray G. Brady, Slippery Rock State College, Slippery Rock, Pennsylvania.

It follows from the identity $F_{2n} = F_n L_n$ that the series are identical. Since

$$\lim_{n \to \infty} \left( \frac{F_{2n}}{r^{2n}} \right) = 1$$

where

$$r = \frac{(1 + \sqrt{5})}{2} > 1,$$

the series converge.


A NONHOMOGENEOUS RECURSION

B-222 Proposed by V. E. Hoggatt, Jr., San Jose State University, San Jose, California.

Find a formula for $K_n$ where $K_1 = 1$ and

$$K_{n+1} = (K_1 + K_2 + \cdots + K_n) + F_{2n+1}.$$


Reducing the subscript $n + 1$ to $n$ in the given recursion, we have $K_n = (K_1 + K_2 + \cdots + K_{n-1}) + F_{2n-1}$. Subtracting from corresponding sides of the original gives us

$$K_{n+1} - K_n = K_n + F_{2n+1} - F_{2n-1} = K_n + F_{2n}$$

or

$$K_{n+1} = 2K_n + F_{2n}.$$

Then $K_1 = 1$, $K_2 = 2\cdot1 + 1$, $K_3 = 2^2\cdot1 + 2\cdot1 + 3$, and generally

$$K_n = 2^{n-1} + (2^{n-2}F_2) + 2^{n-3}F_4 + \cdots + 2F_{2n-4} + F_{2n-2}.$$

Using a result of Herta T. Freitag (Fibonacci Quarterly, Vol. 8, No. 5, p. 344), we have

$$K_n = F_{2n+1} - 2^{n-1}.$$

Also solved by Paul S. Bruckman, L. Carlitz, Herta T. Freitag, Graham Lord, David Zeitlin, Gregory Wulczyn, and the Proposer.

FORMIDABLE ARITHMETIC

B-223 Proposed by Edgar Karst, University of Arizona, Tucson, Arizona.

Find a solution of

$$x^y + (x + 3)^y - (x + 4)^y = u^y + (u + 3)^y - (u + 4)^y.$$
in the form
\[ x = F_m, \quad y = F_n, \quad u = L_4, \quad \text{and} \quad v = L_3. \]

**Solution by the Proposer.**

A solution is
\[ x = 13 = F_7, \quad y = 5 = F_5, \quad u = 7 = L_4, \quad \text{and} \quad v = 3 = L_2. \]

**QUADRATIC NONRESIDUES**

**B-224** Proposed by Lawrence Somer, Champaign, Illinois

Let \( m \) be a fixed positive integer. Prove that no term in the sequence \( F_1, F_3, F_5, F_7, \ldots \) is divisible by \( 4m - 1 \).

**Solution by L. Carlitz, Duke University, Durham, North Carolina.**

Since
\[ 5F_{2n+1}^2 - 4 = L_{2n+1}^2, \]

it would follow from
\[ F_{2n+1} \equiv 0 \pmod{4m - 1} \]

that
\[ L_{2n+1}^2 \equiv -4 \pmod{4m - 1}. \]

This implies the solvability of the congruence
\[ x^2 \equiv -1 \pmod{4m - 1} \]

which is impossible.

*Also solved by Paul S. Bruckman, Graham Lord, and the Proposer.*

**STILL UNCHARACTERIZED SEQUENCES**

**B-225** Proposed by John Ivié, Berkeley, California.

Let \( a_0, \ldots, a_{j-1} \) be constants and let \( \{f_n\} \) be a sequence of integers satisfying
\[ f_{n+j} = a_{j-1}f_{n+j-1} + a_{j-2}f_{n+j-2} + \cdots + a_0f_n; \quad n = 0, 1, 2, \ldots. \]

Find a necessary and sufficient condition for \( \{f_n\} \) to have the property that every integer \( m \) is an exact divisor of some \( f_k \).

**EDITORIAL NOTE:** A necessary and sufficient condition that \( m \) divides some \( f_k \) is that \( m \) divides some \( f_k \) for \( 1 \leq k \leq m \), for every \( m \).