A CONSTRUCTIVE UNIQUENESS THEOREM ON REPRESENTING INTEGERS

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Let F_n be the n^{th} Fibonacci number, i.e., $F_1 = 1$, $F_2 = 2$, and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 3$. It is well known [1] that every integer $N \geq 1$ has a unique representation

$$(1) \qquad N = F_{i_1} + F_{i_2} + \cdots + F_{i_\alpha}$$

such that

$$(2) \qquad i_1 \geq 1, \qquad i_j - i_{j-1} \geq 2 \quad \text{for} \quad j \geq 2 .$$

Conversely, if for all the integers $N \geq 1$,

$$(3a) \qquad N = a_{i_1} + a_{i_2} + \cdots + a_{i_\alpha}$$

is unique under (2), then $a_j = F_j$ for all j, i.e., the uniqueness of (1) under (2) characterizes the Fibonacci sequence. Generalizing this theorem, I shall prove in the present note that at most one increasing sequence can represent uniquely all the integers $N \geq 1$ as sums of its elements under a given constraint and I shall give a combinatorial formula for this only possible sequence.

Let $e_1, e_2, \cdots$ be non-negative integers and let C be a property which classifies each finite ordered set $(e_1, e_2, \cdots, e_n)$ into one of the two categories, those which possess C and those which do not. Denote by $C(e)$ the collection of all the sequences satisfying C.

Let $a_1 < a_2 < \cdots$ be positive integers. Assume that every integer $N > 1$ has a unique representation in the form

$$(3) \qquad N = \sum e_i a_i , \qquad \{e_i\} \in C(e)$$

and it is further assumed that

$$(4) \qquad \text{if} \quad a_n \leq N < a_{n+1} \quad \text{then} \quad e_n \neq 0 .$$

My aim is to prove the following

Theorem. If the property C is expressible independently of $a_1, a_2, \cdots$ then there is at most one sequence $0 < a_1 < a_2 < \cdots$ for which the representation (3) and (4) is unique. In this case, $a_1 = 1$ and for $n > 1$,

$$(5) \qquad a_{n+1} = 1 + \sum_{d=1}^{n} k(n, d, C) ,$$

where $k(n,d,C)$ is the number of n-vectors $(e_1, e_2, \cdots, e_n)$ satisfying C and such that exactly d of its coordinates differ from zero.

Before giving its proof, I wish to make some remarks on the theorem itself and on its applications. First of all, I want to emphasize the second part of the theorem, namely, that the sequence a_n is explicitly determined. In several concrete cases when the structure of $C(e)$ is given, the uniqueness of $\{a_i\}$ can be shown by a simple argument but (5) is not obvious even in these cases, and for a general $C(e)$ the usual argument for the uniqueness, too, seems to be very complicated, if it works at all, since several cases should be distinguished. The formula (5) is very useful at obtaining information on the number of non-zero terms in (3) even if no explicit formula for $k(n,d,C)$ is known. As an example, I mention a recent work of A. Oppenheim. Generalizing (1), he considered the following problem (personal communication). Let k_j, $j \geq 1$ be given positive integers and assume that (3a) is unique under the assumption that the first non-zero term in $i_j - i_{j-1} - k_1$, $i_{j+1} - i_j - k_2, \cdots$ is positive for all $j \geq 2$. In our notations it means that $C(e)$ consists of all $(e_1, e_2, \cdots, e_n)$, $n > 2$, where e_j is either zero or one and if the gap between the j^{th} and the $(j+1)^{st}$ one in $(e_1, e_2, \cdots, e_n)$ is m_j, then for all j, $m_j - k_1$, $m_{j+1} - k_2, \cdots$ has the property that the first non-zero term is positive. A. Oppenheim determined the sequences k_j for which such a representation exists (to be published). In our approach we obtain a construction for the corresponding a's though here $k(n,d,C)$ is a complicated expression. However, this combinatorial function has already been investigated in much details since it has close relations to β-expansions, see [3], which has a wide literature. Two special cases of this problem of Oppenheim, namely, when all $k_j = 2$, or more generally, when for all j, $k_j = k$, have been investigated earlier. The case $k_j = 2$ for all j is simply the condition (2), hence the corresponding sequence a_j is the Fibonacci sequence and the formula (5) gives back its relation to the Pascal triangle. When for all j, $k_j = k$, we get the generalized Fibonacci sequence introduced by Daykin [1], the original argument for the validity of (5) being fairly complicated even for this simple case. In my recent paper [2], I obtained (5) for the generalized Fibonacci numbers, and actually that investigation led to the discovery of the short proof of this general theorem, which now follows.

Proof. First of all, note that (3) and (4) imply that there is a one-to-one correspondence between the integers $1 < N < a_{n+1}$ and the set of n-vectors $(e_1, e_2, \cdots, e_n)$ $C(e)$. As a matter of fact, in view of (4), for any $(e_1, e_2, \cdots, e_n)$ belonging to $C(e)$,

$$e_1a_1 + e_2a_2 + \cdots + e_na_n < a_{n+1} \tag{6}$$

namely, if the reversal of the inequality (6) apply, then, putting M for the left-hand side of (6), in view of (4), M would have a representation with an a_j, $j \geq n+1$, taking part, which by the definition of M, contradicts the uniqueness of (3). The converse of the one-to-one correspondence in question is obvious by (4).

From this observation the proof is easily completed. Cancel those terms in (3) for which $e_j = 0$, hence (3) determines a function $d(N)$, the number of non-zero terms in (3). Since

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