A CONSTRUCTIVE UNIQUENESS THEOREM ON REPRESENTING INTEGERS

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Let $F_n$ be the $n$th Fibonacci number, i.e., $F_1 = 1$, $F_2 = 2$, and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 3$. It is well known [1] that every integer $N \geq 1$ has a unique representation

(1) $N = F_{i_1} + F_{i_2} + \cdots + F_{i_k}$

such that

(2) $i_1 \geq 1$, $i_j - i_{j-1} \geq 2$ for $j \geq 2$.

Conversely, if for all the integers $N \geq 1$,

(3a) $N = a_{i_1} + a_{i_2} + \cdots + a_{i_k}$

is unique under (2), then $a_j = F_j$ for all $j$, i.e., the uniqueness of (1) under (2) characterizes the Fibonacci sequence. Generalizing this theorem, I shall prove in the present note that at most one increasing sequence can represent uniquely all the integers $N \geq 1$ as sums of its elements under a given constraint and I shall give a combinatorial formula for this only possible sequence.

Let $e_1, e_2, \cdots$ be non-negative integers and let $C$ be a property which classifies each finite ordered set $(e_1, e_2, \cdots, e_n)$ into one of the two categories, those which possess $C$ and those which do not. Denote by $C(e)$ the collection of all the sequences satisfying $C$.

Let $a_1 < a_2 < \cdots$ be positive integers. Assume that every integer $N > 1$ has a unique representation in the form

(3) $N = \sum e_1 a_1$, \hspace{1cm} \{e_1\} \in C(e)$

and it is further assumed that

(4) if $a_n \leq N < a_{n+1}$ then $e_n \neq 0$.

My aim is to prove the following

Theorem. If the property $C$ is expressible independently of $a_1, a_2, \cdots$ then there is at most one sequence $0 < a_1 < a_2 < \cdots$ for which the representation (3) and (4) is unique. In this case, $a_1 = 1$ and for $n > 1$,

(5) $a_{n+1} = 1 + \sum_{d=1}^{n} k(n, d, C)$,
where \( k(n, d, C) \) is the number of \( n \)-vectors \( (e_1, e_2, \ldots, e_n) \) satisfying \( C \) and such that exactly \( d \) of its coordinates differ from zero.

Before giving its proof, I wish to make some remarks on the theorem itself and on its applications. First of all, I want to emphasize the second part of the theorem, namely, that the sequence \( a_n \) is explicitly determined. In several concrete cases when the structure of \( C(e) \) is given, the uniqueness of \( \{a_j\} \) can be shown by a simple argument but \( 5 \) is not obvious even in these cases, and for a general \( C(e) \) the usual argument for the uniqueness, too, seems to be very complicated, if it works at all, since several cases should be distinguished. The formula \( 5 \) is very useful at obtaining information on the number of non-zero terms in \( 3 \) even if no explicit formula for \( k(n, d, C) \) is known. As an example, I mention a recent work of A. Oppenheim. Generalizing \( 1 \), he considered the following problem (personal communication). Let \( k_j, j \geq 1 \) be given positive integers and assume that \( 3a \) is unique under the assumption that the first non-zero term in \( l_j = l_{j-1} - k_1, l_{j+1} - l_j - k_2, \ldots \) is positive for all \( j \geq 2 \). In our notations it means that \( C(e) \) consists of all \( (e_1, e_2, \ldots, e_n) \), \( n \geq 2 \), where \( e_j \) is either zero or one and if the gap between the \( j^{th} \) and the \((j + 1)^{st} \) one in \( (e_1, e_2, \ldots, e_n) \) is \( m_j \), then for all \( j, m_j - k_1, m_{j+1} - k_2, \ldots \) the first non-zero term is positive. A. Oppenheim determined the sequences \( k_j \) for which such a representation exists (to be published). In our approach we obtain a construction for the corresponding \( a_j \)'s though here \( k(n, d, C) \) is a complicated expression. However, this combinatorial function has already been investigated in much details since it has close relations to \( \beta \)-expansions, see [3], which has a wide literature. Two special cases of this problem of Oppenheim, namely, when all \( k_j = 2 \), or more generally, when for all \( j, k_j = k \), have been investigated earlier. The case \( k_j = 2 \) for all \( j \) is simply the condition \( 2 \), hence the corresponding sequence \( a_j \) is the Fibonacci sequence and the formula \( 5 \) gives back its relation to the Pascal triangle. When for all \( j, k_j = k \), we get the generalized Fibonacci sequence introduced by Daykin [1], the original argument for the validity of \( 5 \) being fairly complicated even for this simple case. In my recent paper [2], I obtained \( 5 \) for the generalized Fibonacci numbers, and actually that investigation led to the discovery of the short proof of this general theorem, which now follows.

**Proof.** First of all, note that \( 3 \) and \( 4 \) imply that there is a one-to-one correspondence between the integers \( 1 \leq N \leq a_{n+1} \) and the set of \( n \)-vectors \( (e_1, e_2, \ldots, e_n) \) \( C(e) \). As a matter of fact, in view of \( 4 \), for any \( (e_1, e_2, \ldots, e_n) \) belonging to \( C(e) \),

\[
e_1a_1 + e_2a_2 + \cdots + e_na_n < a_{n+1}
\]

namely, if the reversal of the inequality \( 6 \) apply, then, putting \( M \) for the left-hand side of \( 6 \), in view of \( 4 \), \( M \) would have a representation with an \( a_j \), \( j \geq n + 1 \), taking part, which by the definition of \( M \), contradicts the uniqueness of \( 3 \). The converse of the one-to-one correspondence in question is obvious by \( 4 \).

From this observation the proof is easily completed. Cancel those terms in \( 3 \) for which \( e_j = 0 \), hence \( 3 \) determines a function \( d(N) \), the number of non-zero terms in \( 3 \). Since [Continued on page 598.]