A NEW GREATEST COMMON DIVISOR PROPERTY OF THE BINOMIAL COEFFICIENTS

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1. INTRODUCTION

The chief object of this paper is to announce the following:

Conjecture. Let k and n be any integers with $0 \le k \le n$, and

$$\binom{n}{k} = n!/k!\,(n-k)!$$

be the ordinary binomial coefficients. Then

$$\text{(1.1)} \qquad \gcd\left\{\binom{n-1}{k}, \binom{n}{k-1}, \binom{n+1}{k+1}\right\} = \gcd\left\{\binom{n-1}{k-1}, \binom{n}{k+1}, \binom{n+1}{k}\right\} .$$

The consideration of this matter was prompted by a result due to Hoggatt and Hansell [3] which is that

$$\text{(1.2)} \qquad \binom{n-1}{k}\binom{n}{k-1}\binom{n+1}{k+1} = \binom{n-1}{k-1}\binom{n}{k+1}\binom{n+1}{k} .$$

The six coefficients involved form a hexagonal pattern around $\binom{n}{k}$ in the usual Pascal triangle display. See the diagram in [1] where I called (1.2) a Star of David Property. The new conjecture gives a new Star of David property. What is more, I also conjecture that (1.1) holds for Fibonomial coefficients where n! is replaced by

$$[n]! = F_n F_{n-1} \cdots F_2 F_1 , \quad [0]! = 1 ,$$

with

$$F_{n+1} = F_n + F_{n-1}, \qquad F_0 = 0, \qquad F_1 = 1 ,$$

being the ordinary Fibonacci numbers. The manner in which powers of a prime enter as factors of such generalized coefficients suggests that there are many other arrays in which the new arithmetic Star of David property holds. We shall also exhibit some entirely novel pseudo-binomial coefficient arrays where the conjecture holds. It would be of great interest to establish necessary and/or sufficient conditions for the new conjecture. I am certain the conjecture is correct but hesitate to publish a proof as I believe my original proof has a flaw. Computational results will be exhibited here as evidence.

2. EVIDENCE

Table 1 below shows the situation for 21 rows of the Pascal triangle. Shown here is

$$\gcd\left\{\binom{n-1}{k},\ \binom{n}{k-1},\ \binom{n+1}{k+1}\right\},$$

for $0 \le k \le n/2$. In every case the value is identical with

$$\gcd\left\{\binom{n-1}{k-1},\ \binom{n}{k+1},\ \binom{n+1}{k}\right\}.$$

Spot checks for dozens of other values have failed to turn up any counterexample. In working with numerical examples, it is convenient to draw the Pascal triangle in the usual manner as

$$\begin{array}{ccccccccccccccc}
 & & & & & & & 1 & & & & & & & \\
 & & & & & & 1 & & 1 & & & & & & \\
 & & & & & 1 & & 2 & & 1 & & & & & \\
 & & & & 1 & & 3 & & 3 & & 1 & & & & \\
 & & & 1 & & 2^2 & & 2\cdot 3 & & 2^2 & & 1 & & & \\
 & & 1 & & 5 & & 2\cdot 5 & & 2\cdot 5 & & 5 & & 1 & & \\
 & 1 & & 2\cdot 3 & & 3\cdot 5 & & 2^2\cdot 5 & & 3\cdot 5 & & 2\cdot 3 & & 1 & \\
1 & & 7 & & 3\cdot 7 & & 5\cdot 7 & & 5\cdot 7 & & 3\cdot 7 & & 7 & & 1
\end{array}$$

but in factored form. The way in which the primes appear suggests both (1.1) and (1.2). Because of the recurrence relation governing formation of the binomial coefficients (and the same principle applies to the Fibonomial coefficients) the occurrence of prime factors forms a triangular pattern. Thus, if

$$p^a \,\Big|\, \binom{n}{k} \quad\text{and}\quad p^b \,\Big|\, \binom{n}{k-1},$$

then

$$p^c \,\Big|\, \binom{n+1}{k}$$

where $c = \min(a,b)$. But c may be larger!

Let us denote the set of coefficients

$$\left\{\binom{n-1}{k},\ \binom{n}{k-1},\ \binom{n+1}{k+1}\right\}$$

by $\triangleleft$ and the set

$$\left\{\binom{n-1}{k-1}, \binom{n}{k+1}, \binom{n+1}{k}\right\}$$

by $\triangleright$, or more generally, we may sometimes use this suggestive notation for the corresponding sets in any general array. If we must be explicit we can write $\triangleleft_{n,k}$ and $\triangleright_{n,k}$, to indicate the values of n and k used. Clearly, if we compute a table of g.c.d. $\triangleleft$ and the table is symmetrical with an entry in the k spot on row n the same as the entry in the n - k spot, then the property (1.1) holds. This is because of the similar symmetry for the Pascal triangle itself. Table 1, therefore, lists g.c.d. $\triangleleft$ for $0 \le k \le n/2$ only. The original table was drawn up on a very large sheet of paper and is not easy to reproduce here.

Table 1

n	0	1	2	3	4	···	k		···	[n/2]	
0	1										
1	1										
2	1	1									
3	1	1									
4	1	1	1								
5	1	1	1								
6	1	1	1	5							
7	1	1	1	1							
8	1	1	1	7	7						
9	1	1	1	2	14						
10	1	1	1	3	6	42					
11	1	1	1	5	3	6					
12	1	1	1	11	11	33	66				
13	1	1	1	1	11	11	33				
14	1	1	1	13	13	143	143	429			
15	1	1	1	7	91	91	143	143			
16	1	1	1	5	7	91	13	143	715		
17	1	1	1	4	4	28	52	26	286		
18	1	1	1	17	68	68	68	442	442	4862	
19	1	1	1	3	51	204	204	102	442	442	
20	1	1	1	19	57	969	3876	1938	646	8398	8398

A result like (1.1) using l.c.m. is in general false. The first simple counter-example is

$$\operatorname{lcm} \triangleleft_{3,1} = \operatorname{lcm}\left\{\binom{2}{1}, \binom{3}{0}, \binom{4}{2}\right\} = \operatorname{lcm}(2, 1, 6) = 6 ,$$

whereas

$$\text{lcm} \triangleright_{3,1} = \text{lcm} \left\{ \binom{2}{0}, \binom{3}{2}, \binom{4}{1} \right\} = \text{lcm}\ (1, 3, 4) = 12 .$$

There are, however, numerous cases where the l. c. m. property does hold.

Except for the first value, it is interesting to note that the sequence of middle numbers in Table 1, i.e., 1, 1, 1, 5, 7, 42, 66, 429, 715, 4862, 8398, ⋯, are alternately Catalan numbers or one-half Catalan numbers. More precisely: let $n \geq 1$. Then

$$\text{(2.1)} \qquad \gcd \left\{ \binom{2n-1}{n}, \binom{2n}{n-1}, \binom{2n+1}{n+1} \right\} = \begin{cases} \binom{2n}{n} \dfrac{1}{n+1}, & 2 \nmid n, \\ \dfrac{1}{2} \binom{2n}{n} \dfrac{1}{n+1}, & 2 \mid n. \end{cases}$$

We omit the proof.

3. THE FIBONOMIAL CASE

The corresponding result for the Fibonomial coefficients to (1.1) is true because these numbers satisfy a recurrence relation similar to that for the ordinary binomial coefficients. We should remark that the same may be said for the Gaussian or q-binomial coefficients. We omit the details of the proof.

To illustrate the relation (1.1) for Fibonomial coefficients, we give in Table 2 some specimen values. The table starts with $n = 6$, the first row where the g. c. d. > 1 for any k.

Table 2

n	0	1	2	3 ⋯		k ⋯	[n/2]
6	1	1	1	5			
7	1	1	1	4			
8	1	1	1	13	52		
9	1	1	1	7	91		
10	1	1	1	17	119	1547	
11	1	1	1	55	187	1309	

Again one finds a formula for Fibonomial Catalan numbers, but it is not as simple as (2.1).

4. PSEUDO-BINOMIAL COEFFICIENTS

Scrutiny of the discussion above for (1.1) shows that the key to the pattern of prime powers lies in the recurrence relation used. However, we may evidently dispense with the recurrence relation and still have (1.1). To illustrate, we offer the array on the following page of pseudo-binomial coefficients.

1

1 1

1 1 1

1 2 2 1

1 3 2·3 3 1

1 5 3·5 3·5 5 1

1 7 5·7 3·5·7 5·7 7 1

1 11 7·11 5·7·11 5·7·11 7·11 11 1

1 13 11·13 7·11·13 5·7·11·13 7·11·13 11·13 13 1

Here we have imposed a perfectly regular pattern of appearance of prime factors. It is easy to see that (1.1) must hold for the pseudo-binomial coefficients P(n,k). A few specimen rows from the g. c. d. triangle are:

5

7 7

11 7·11 11

13 11·13 11·13 13

17 13·17 11·13·17 13·17 17

where we have tabulated the g. c. d. for $3 \le k \le n - 3$ and $6 \le n \le 10$.

It is also evident that the resulting array itself possesses property (1.1), and this may be seen to repeat forever. The l. c. m. of the two sets of coefficients in (1.1) fail to be equal for the pseudo-binomial coefficients for $k = 0$ $(n \ge 2)$, and for $k = 2$ $(n = 5)$, $k = 3$ $(n = 7)$, $k = 4$ $(n = 9)$, etc. We omit a discussion of the precise behavior of the least common multiples, but it is clearly a matter to be investigated. I have been unable to find an array in which the g. c. d. property and l. c. m. property both hold always. Even l. c. m. arrays are hard to come by.

In contrast to the Pascal triangle and the Fibonomial triangle, the array of pseudo-binomial coefficients does not have the property (1.2) of Hoggatt-Hansell.

Here is still another pseudo-binomial array having the Star of David property (1.1):

1

1 1

1 2 1

1 3 3 1

1 7 3·7 7 1

1 1 7 7 1 1

1 2^3 2 2·7 2 2^3 1

1 5^2 $2 \cdot 5^2$ $2 \cdot 5^2$ $2 \cdot 5^2$ $2 \cdot 5^2$ 5^2 1

One may easily extend such a triangle in an infinity of ways.

These are the types of general array suggested by our work, arrays in which the entry of primes occurs in carefully delineated triangles. The most general such triangle has not been written out.

5. MULTINOMIALS

It is, of course, tempting to go further. In [1], [2], [4] will be found methods for finding equal products of any number of binomial and multinomial coefficients in general. Whenever a triangle pattern of prime entry appears, one suspects that interesting g. c. d. and l. c. m. properties will hold in certain cases. Computer calculations would be very useful to make further conjectures, but already I have checked numerous cases and found interesting results. When one realizes that Scharff, Rine, and Gould [2] have found relations such as

$$\binom{n+2}{k-1}\binom{n-3}{k}\binom{n+3}{k+1}\binom{n-2}{k-2}\binom{n+1}{k+2}\binom{n}{k-3}\binom{n-1}{k+3}$$

$$= \binom{n-2}{k-1}\binom{n+3}{k}\binom{n-3}{k+1}\binom{n+1}{k-2}\binom{n+2}{k+2}\binom{n-1}{k-3}\binom{n}{k+3} ,$$

it becomes clear that there is much more to be investigated. When, for example, are the g. c. d.'s of the above sets of seven coefficients equal? Not in general, as examples are easily shown to the contrary. A computer can easily generate as many tables of this sort as needed. We should remark that the detailed computer print-out in [2] will be deposited in the Fibonacci Bibliographical Center for reference.

In [1] I pointed out that (1.2) generalizes to

$$\binom{n-a}{k}\binom{n}{k-a}\binom{n+n}{k+a} = \binom{n-a}{k-a}\binom{n}{k+a}\binom{n+a}{k}$$

and it is tempting to see if the g. c. d. property holds here. A simple counter-example, $n = 8$, $k = 3$, $a = 2$ suffices to show that the g. c. d. Star of David property does not hold in general here. Again, however, abundant true examples exist.

ADDENDUM

Property (1.1) was first noted by me around December 1971. Since writing the present paper (1.1) was mentioned to Hoggatt (telephone call, August 3, 1972), and I have now heard from him (telephone call August 7) that he and A. P. Hillman [5] have proved conjecture (1.1) as well as for the Fibonomial case and for arrays in general where certain recurrences hold, The method is one due to Hillman based on iteration and the recurrence. Clearly we are at the opening of a new chapter in the discovery of interesting arithmetic properties of arrays of numbers.

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