SOME COMBINATORIAL IDENTITIES OF BRUCKMAN
A SYSTEMATIC TREATMENT WITH RELATION TO THE OLDER LITERATURE

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Bruckman [4] has made a study of some properties of numbers A_n defined by the power series expansion

(1) $$f(x) = (1 - x)^{-1}(1 + x)^{-1/2} = \sum_{n=0}^{\infty} A_n x^n .$$

In some cases, for convenience, he uses the modified notation

(2) $$B_n = 2^n n! A_n .$$

By use of the binomial theorem he found that

(3) $$A_n = \sum_{k=0}^{n} (-1)^k \binom{2k}{k} 2^{-2k} .$$

Then by means of an exponential integral he was able to show that

(4) $$A_n = 2^{-2n}(2n + 1)\binom{2n}{n}\sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{2^k}{2k + 1} .$$

The A's satisfy the second-order recurrence relation

(5) $$2n A_n = A_{n-1} + (2n - 1)A_{n-2} , \qquad A_0 = 1, \ A_1 = 1/2 .$$

Using recurrence relations and differential equations, Bruckman obtained the following elegant formula

(6) $$A_n^2 = (2n + 1)\binom{2n}{n} 2^{-2n} \sum_{k=0}^{n} (-1)^{n-k} \binom{2k}{k} 2^{-2k} \frac{1}{2n + 1 - 2k} .$$

Bruckman proves this interesting formula by showing that

$$\frac{\text{Arctan } x}{\sqrt{1 - x^2}} = \sum_{n=0}^{\infty} B_n^2 \frac{x^{2n+1}}{(2n+1)!}, \tag{7}$$

while, on the other hand, it is easy to multiply the series for Arctan x and $(1 - x^2)^{-1/2}$ together directly, and the result is (6).

I believe that formula (6) is the most interesting formula given in [4], and it does not appear in any readily accessible source. A direct proof of (6) by squaring (3) is not exactly trivial. The other relations in [4] are not really new, and far more general expansions have been considered in the older literature. However, it is hard to name a single source where all such expansions have been systematically generated. In the work below we shall obtain variant forms and expansions and in passing show that the numbers A_n are special cases of numbers studied by Cauchy [5], Chessin [6, 7], Perna [10], and Graver [9]. Some of the power series expansions are summarized in Adams and Hippisley [1] who also cite other related sums. Since our motive is partly pedagogical, we give considerable detail in some of the proofs below. We end by stating a difficult RESEARCH PROBLEM.

Free use will be made of some elementary identities, such as

$$\binom{-1/2}{k} = (-1)^k \binom{2k}{k} 2^{-2k}, \tag{8}$$

which follow from the polynomial definition of the binomial coefficient

$$\binom{x}{k} = \frac{x(x-1)\cdots(x-k+1)}{k!}, \qquad \binom{x}{0} = 1.$$

For example, we also have

$$\binom{-x}{k} = (-1)^k \binom{x+k-1}{k}, \qquad x = \text{any real number.} \tag{9}$$

We shall use the older notation $((x^n))F(x)$ to denote the coefficient of x^n in the power series expansion of $F(x)$.

We now summarize the main formulas proved and discussed in the present paper:

$$((x^n))(1-x)^{-1}(1+x)^{-1/2} = \sum_{k=0}^{n} \binom{-1/2}{k} = \sum_{k=0}^{n} (-1)^k \binom{2k}{k} 2^{-2k} = A_n. \tag{10}$$

This is just Bruckman's first result with a variation by means of (8).

$$((x^n))(1-x)^{-1/2}(1-x^2)^{-1/2} = \sum_{k=0}^{[n/2]} (-1)^k \binom{-1/2}{k}\binom{-1/2}{n-2k} = A_n. \tag{11}$$

(12)
$$((x^n))\,(1-x)^{-3/2}\left(1+\frac{2x}{1-x}\right)^{-1/2} = \sum_{k=0}^{n}(-1)^k\binom{2k}{k}\binom{n+1/2}{n-k}2^{-k} = A_n$$
$$= 2^{-2n}(2n+1)\binom{2n}{n}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{2k}{2k+1}\;.$$

(13)
$$((x^n))\,(1+x)^{-3/2}\left(1-\frac{2x}{1+x}\right)^{-1}$$
$$= 2^{-2n}\binom{2n}{n}\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}\binom{2k}{k}^{-1}2^{3k}\frac{2n+1}{2k+1} = A_n\;.$$

(14)
$$e^{x^2/2}\int_0^x e^{-u^2}du = \sum_{n=0}^{\infty}\frac{x^{2n+1}}{n!\,2^n}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{2^k}{2k+1}$$
$$= \sum_{n=0}^{\infty}\frac{x^{2n+1}}{(2n+1)!}\,2^n n!\,A_n = \sum_{n=0}^{\infty}\frac{x^{2n+1}}{(2n+1)!}\,B_n\;.$$

This is just relation (22) in [4].

(15)
$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\binom{2k}{k}^{-1}2^{2k}\frac{A_k}{2k+1} = \frac{2^n}{2n+1}\;.$$

(16)
$$\sum_{k=0}^{n}\binom{n}{k}\binom{2k}{k}^{-1}2^{2k}\frac{A_k}{2k+1} = \frac{2^{3n}}{2n+1}\binom{2n}{n}^{-1}\;.$$

(17)
$$\sum_{k=0}^{[n/2]}\binom{n}{2k}\binom{4k}{2k}^{-1}2^{4k}\frac{A_{2k}}{4k+1} = \frac{2^{n-1}}{2n+1}\left\{1+2^{2n}\binom{2n}{n}^{-1}\right\}\;.$$

Relation (17) follows by adding (15) and (16) together so that odd-index terms cancel. Subtracting (15) from (16) yields a similar formula involving A_{2k+1}.

(18)
$$\sum_{n=0}^{\infty}2^{2n+1}\binom{2n}{n}^{-1}\frac{A_n}{2n+1}\cdot\frac{x^{2n+1}}{(2-x^2)^{n+1}} = \frac{\operatorname{Arcsin} x}{\sqrt{1-x^2}}\;.$$

(19) $$\operatorname{Arctan} x = \sum_{n=0}^{\infty} 2^{2n+1}\binom{2n}{n}^{-1} \frac{A_n}{2n+1} \cdot \frac{x^{2n+1}}{(2+x^2)^{n+1}} .$$

(20) $$B_n = \frac{(2n+1)!}{2^n\, n!\, \sqrt{2}} \int_0^{\sqrt{2}} (1-t^2)^n \, dt .$$

This is relation (26) in [4].

(21) $$B_n = \frac{(2n+1)!}{2^n\, n!} \int_0^1 (1-2u^2)^n \, du .$$

This relation follows from (20) by the change of variable $t = u\sqrt{2}$.

(22) $$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} \sum_{k=0}^{n} \binom{-1/2}{n-k} \frac{2n+1}{2k+1} = \frac{\operatorname{Arctan} x}{\sqrt{1-x^2}} .$$

(23) $$\left\{ \sum_{k=0}^{n} \binom{-1/2}{k} \right\}^2 = A_n^2 = \binom{-1/2}{n} \sum_{k=0}^{n} \binom{-1/2}{n-k} \frac{2n+1}{2k+1} .$$

This is an equivalent formulation of Bruckman's formula (6) above

(24) $$\sum_{j=0}^{n} \binom{-1/2}{j} \cdot \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} 2^k \frac{2n+1}{2k+1} = \sum_{k=0}^{n} \binom{-1/2}{n-k} \frac{2n+1}{2k+1} .$$

This is another equivalent formulation of (6).

(25) $$\sum_{n=0}^{\infty} t^n \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{x^k}{2k+1} = \frac{1}{1-t} \sum_{k=0}^{\infty} \frac{1}{2k+1} \left(\frac{-xt}{1-t} \right)^k$$
$$= S(x,t) = \frac{1}{1-t} \cdot \frac{\operatorname{Arctan} z}{z} ,$$

where

$$z^2 = xt/(1-t) .$$

For $x = 2$, this may be specialized to involve A_n or B_n, whence

(26)
$$S(2,t) = \sum_{n=0}^{\infty} t^n \frac{2^n n!}{(2n+1)!} B_n .$$

(27)
$$A_n^2 = (2n+1)^2 \binom{-1/2}{n}^2 \sum_{k=0}^{n} (-1)^k 2^k \sum_{j=0}^{k} \binom{n}{j}\binom{n}{k-j} \frac{1}{(2j+1)(2k-2j+1)} .$$

(28)
$$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} A_n = \sum_{k=0}^{\infty} x^{k+1} \frac{J_k(x)}{(2k+1)k!} ,$$

where $J_k(x)$ is the ordinary Bessel function.

The power series in this paper are treated as formal power series, without regard to regions of convergence. The algebra of such formal power series is developed in Niven's paper [11]. Convergence information for the various series could be developed, but we shall omit this.

The functions expanded in (10), (11), (12), (13) are all identical with Bruckman's definition in (1). The proof of (10) is trivial, being a direct application of the binomial theorem and Cauchy product of series.

Here are details of a proof of (11):

$$(1-x)^{-1/2}(1-x^2)^{-1/2} = \sum_{j=0}^{\infty} (-1)^j \binom{-1/2}{j} x^j \cdot \sum_{k=0}^{\infty} (-1)^k \binom{-1/2}{k} x^{2k}$$
$$= \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} (-1)^{j+k} \binom{-1/2}{j}\binom{-1/2}{k} x^{j+2k} .$$

In this, let $j = n - 2k$ to obtain the coefficient of x^n. The result is (11).

In proving (12) we first note that the identity, which is (Z.46) in [8],

$$\frac{2n+1}{2k+1}\binom{2n}{n}\binom{n}{k} 2^{-2n} = \binom{2k}{k}\binom{n+1/2}{n-k} 2^{-2k} , \quad 0 \le k \le n ,$$

can be obtained from the polynomial definition of $\binom{x}{n}$ just as (8) is found. Thus we have only to prove the first form of (12), which can be done as follows:

$$\sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{2k}{k}\binom{n+1/2}{n-k} 2^{-k} (-1)^k x^n$$

$$= \sum_{k=0}^{\infty} \binom{2k}{k} 2^{-k}(-1)^k \sum_{n=k}^{\infty} \binom{n + 1/2}{n - k} x^n$$

$$= \sum_{k=0}^{\infty} (-1)^k \binom{2k}{k} 2^{-k}x^k \sum_{n=0}^{\infty} \binom{n + k + 1/2}{n} x^n ,$$

using the substitution of $n + k$ for n,

$$= \sum_{k=0}^{\infty} (-1)^k \binom{2k}{k} 2^{-k}x^k \sum_{n=0}^{\infty} (-1)^n \binom{-3/2 - k}{n} x^n ,$$

by use of (9),

$$= \sum_{k=0}^{\infty} (-1)^k \binom{2k}{k} 2^{-k}x^k(1 - x)^{-3/2-k} ,$$

by the Binomial theorem,

$$= (1 - x)^{-3/2} \sum_{k=0}^{\infty} \binom{-1/2}{k} \left(\frac{2x}{1 - x}\right)^k ,$$

using (8),

$$= (1 - x)^{-3/2} \left(1 + \frac{2x}{1 - x}\right)^{-1/2} = (1 - x)^{-1}(1 + x)^{-1/2} .$$

The somewhat similar proof of (13) runs as follows:

$$\sum_{n=0}^{\infty} x^n \binom{2n}{n} 2^{-2n}(2n + 1) \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} \binom{2k}{k}^{-1} 2^{3k} \frac{1}{2k + 1}$$

$$= \sum_{k=0}^{\infty} \sum_{n=k}^{\infty} x^n \binom{n + 1/2}{n - k} 2^k(-1)^{n-k} ,$$

using (Z.46) in [8],

$$= \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} x^{n+k} \binom{n + k + 1/2}{n} 2^k(-1)^n$$

$$= \sum_{k=0}^{\infty} (2x)^k \sum_{n=0}^{\infty} (-1)^n \binom{-3/2 - k}{n} x^n (-1)^n ,$$

using (9),

$$= \sum_{k=0}^{\infty} (2x)^k (1 + x)^{-3/2-k} = (1 + x)^{-3/2} \sum_{k=0}^{\infty} \left(\frac{2x}{1 + x}\right)^k$$

$$= (1 + x)^{-3/2} \left(1 - \frac{2x}{1 + x}\right)^{-1} = (1 + x)^{-1/2}(1 - x)^{-1} .$$

We have said that the expansions which we consider are special cases of other known expansions. To illustrate this, we note the formula

$$\text{(29)} \qquad \sum_{k=0}^{n} (-1)^k \binom{x}{k}\binom{y}{n - k} = \sum_{k=0}^{[n/2]} (-1)^k \binom{x}{k}\binom{y - x}{n - 2k} ,$$

valid for all real or complex x and y. This is formula (3.31) in [8]. In this formula, let $x = -1/2$, $y = -1$, and we obtain at once

$$\sum_{k=0}^{[n/2]} (-1)^k \binom{-1/2}{k}\binom{-1/2}{n - 2k} = \sum_{k=0}^{n} (-1)^k \binom{-1/2}{k}\binom{-1}{n - k} ,$$

but by (9) we have

$$\binom{-1}{j} = (-1)^j ,$$

so that we have proved the equivalence of (10) and (11) this way.

Again formula (1.9) in [8] is

$$\text{(30)} \qquad \sum_{k=0}^{n} \binom{x}{k} y^k = \sum_{k=0}^{n} \binom{n - x}{k} (1 + y)^{n-k}(-y)^k ,$$

valid for all real or complex x and y. Letting $x = -1/2$, it is easy to see that we obtain the equivalence of (13) and (10). Here again we need (Z.46) in [8].

Still another way to prove the equivalence of (13) and (10) is to use formula (1.10) from [8]:

$$\text{(31)} \qquad \sum_{k=0}^{n} \binom{z}{k} x^{n-k} = \sum_{k=0}^{n} \binom{z - k - 1}{n - k} (x + 1)^k .$$

In this, let $z = -1/2$, $x = 1$, and simplify. This time we need the identity

$$\binom{-1/2 - k}{n - k}\binom{2k}{k} = (-1)^{n-k}\binom{2n}{n}\binom{n}{k}2^{2k-2n}, \qquad 0 \le k \le n,$$

which is easily proved from the polynomial definition of $\binom{x}{n}$.

A direct proof of (14) is as follows:

$$e^{x^2/2}\int_0^x e^{-u^2}du = e^{x^2/2}\int_0^x \sum_{k=0}^{\infty}(-1)^k \frac{u^{2k}}{k!}\,du$$

$$= \sum_{n=0}^{\infty}\frac{x^{2n}}{2^n\, n!}\cdot\sum_{k=0}^{\infty}(-1)^k \frac{x^{2k+1}}{(2k+1)k!}$$

$$= \sum_{k=0}^{\infty}\sum_{n=0}^{\infty}(-1)^k \frac{x^{2n+2k+1}}{2^n\, n!\,(2k+1)k!}$$

$$= \sum_{k=0}^{\infty}\sum_{n=k}^{\infty}(-1)^k \frac{x^{2n+1}}{2^{n-k}(n-k)!\,(2k+1)k!},$$

replacing n by $n - k$,

$$= \sum_{n=0}^{\infty}\frac{x^{2n+1}}{n!\,2^n}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{2^k}{2k+1},$$

as desired to show.

A variant of (14) involving Bessel functions is derived as follows, and is formula (28):

$$\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{(2n+1)!}\,2^{-2n}\,\frac{(2n+1)!}{n!^2 2^{2n}}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{2^k}{2k+1}$$

$$= x\sum_{k=0}^{\infty}(-1)^k\frac{2^k}{2k+1}\sum_{n=k}^{\infty}(-1)^n\left(\frac{x}{2}\right)^{2n}\frac{1}{n!\,(n-k)!\,k!}$$

$$= x\sum_{k=0}^{\infty}\frac{2^k}{(2k+1)k!}\left(\frac{x}{2}\right)^{2k}\sum_{n=0}^{\infty}(-1)^n\left(\frac{x}{2}\right)^{2n}\frac{1}{(n+k)!\,k!}$$

$$= x\sum_{k=0}^{\infty}\frac{2^k}{(2k+1)k!}\left(\frac{x}{2}\right)^{2k}\left(\frac{x}{2}\right)^{-k}J_k(x) = \sum_{k=0}^{\infty}x^{k+1}\frac{J_k(x)}{(2k+1)k!}.$$

Relation (15) is nothing but the inversion of (12), and relation (16) the inversion of (13). What is needed to see this is the well known pair of inverse series:

$$f(n) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} g(k) \tag{32}$$

if and only if

$$g(n) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} f(k) . \tag{33}$$

These in turn depend on nothing deeper than the orthogonality relation

$$\sum_{k=j}^{n} (-1)^{k+j} \binom{n}{k}\binom{k}{j} = \begin{cases} 0, & n \neq j , \\ 1, & n = j , \end{cases}$$

and this is a consequence of the binomial theorem.

To use them, for example, choose $g(k) = 2^k/(2k + 1)$ and then by (12),

$$f(n) = 2^n n!\, B_n /(2n + 1)! .$$

Therefore by (33) we find that (12) inverts to yield (15). Relation (13) inverts to give (16) in a similar way.

Adams and Hippisley [1, p. 122, 6.42-(5.)] give the formula

$$\frac{\operatorname{Arcsin} x}{\sqrt{1 - x^2}} = \sum_{n=0}^{\infty} \frac{2^{2n} n!^2}{(2n + 1)!} x^{2n+1} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n + 1} \binom{-1/2}{n}^{-1} \tag{34}$$

which may be compared with (18) here. Of course, there is also the well known expansion

$$\operatorname{Arcsin} x = \sum_{n=0}^{\infty} (-1)^n \binom{-1/2}{n} \frac{x^{2n+1}}{2n + 1} , \tag{35}$$

which we cite for completeness.

Proof of (18) is obtained in the following way: By (16) we have

$$\frac{2^{2n} n!^2}{(2n + 1)!} = 2^{-n} \sum_{k=0}^{n} \binom{n}{k}\binom{2k}{k}^{-1} 2^{2k} \frac{A_k}{2k + 1} ,$$

then, recalling (34), we have

$$\frac{\text{Arcsin } x}{\sqrt{1-x^2}} = \sum_{n=0}^{\infty} 2^{-n} x^{2n+1} \sum_{k=0}^{n} \binom{n}{k}\binom{2k}{k}^{-1} 2^{2k} \frac{A_k}{2k+1}$$

$$= x \sum_{k=0}^{\infty} 2^{2k} \binom{2k}{k}^{-1} \frac{A_k}{2k+1} \sum_{n=k}^{\infty} \binom{n}{k} x^{2n} 2^{-n}$$

$$= x \sum_{k=0}^{\infty} x^{2k} 2^{k} A_k \binom{2k}{k}^{-1} \frac{1}{2k+1} \left(1 - \frac{x^2}{2}\right)^{-k-1},$$

which reduces to the desired result.

Relation (19) is proved in a similar way from (15), for we have first of all:

$$\text{Arctan } x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = \sum_{n=0}^{\infty} (-1)^n 2^{-n} x^{2n+1} \sum_{k=0}^{n} (-1)^k \binom{n}{k}\binom{2k}{k}^{-1} \frac{2^{2k} A_k}{2k+1}$$

$$= x \sum_{k=0}^{\infty} (-1)^k 2^{2k} \binom{2k}{k}^{-1} \frac{A_k}{2k+1} \sum_{n=k}^{\infty} (-1)^n x^{2n} 2^{-n} \binom{n}{k}$$

$$= x \sum_{k=0}^{\infty} 2^{2k} \binom{2k}{k}^{-1} \frac{A_k}{2k+1} 2^{-k} x^{2k} \sum_{n=0}^{\infty} (-1)^n x^{2n} 2^{-n} \binom{n+k}{k}$$

$$= x \sum_{k=0}^{\infty} 2^{k} \binom{2k}{k}^{-1} \frac{A_k}{2k+1} x^{2k} \left(1 + \frac{x^2}{2}\right)^{-k-1},$$

which reduces as required.

Proof of (22):

$$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} \sum_{k=0}^{n} \binom{-1/2}{n-k} \frac{2n+1}{2k+1}$$

$$= \sum_{n=0}^{\infty} (-1)^n x^{2n+1} \sum_{k=0}^{n} \binom{-1/2}{n-k} \frac{1}{2k+1}$$

$$= \sum_{j=0}^{\infty} (-1)^j \frac{x^{2j+1}}{2j+1} \cdot \sum_{k=0}^{\infty} (-1)^k \binom{-1/2}{k} x^{2k} = \text{Arctan } x \cdot (1 - x^2)^{-1/2}.$$

Relation (23) is found by using (8) in (6), giving

$$A_n^2 = \binom{-1/2}{n} \sum_{k=0}^{n} \binom{-1/2}{n-k} \frac{2n+1}{2k+1},$$

from which the formula follows readily.

Relation (24) is found by writing A_n^2 in (23) as a product of two forms of A_n given by (10) and (12), so that a factor of

$$\binom{-1/2}{n}$$

cancels.

Relation (27) follows from (12) by using the general theorem that

$$\sum_{k=0}^{n} \binom{n}{k} a_k \cdot \sum_{j=0}^{n} \binom{n}{j} b_j = \sum_{k=0}^{n} \sum_{j=0}^{k} \binom{n}{j}\binom{n}{k-j} a_j b_{k-j} \tag{36}$$

for arbitrary a_k's and b_j's.

Relations (23), (24), and (27) are offered as small variations on (6).

An alternative proof of (19) can be given by first noting that

$$\text{Arctan } x = \frac{x}{1+x^2} \sum_{n=0}^{\infty} \frac{2^{2n} n!^2}{(2n+1)!} \left(\frac{x^2}{1+x^2}\right)^n \tag{37}$$

as noted in [1, p. 122, 6.41-(3.)]. One then expands $2^{2n} n!^2/(2n+1)!$ by (16), and upon reduction and use of the binomial theorem we again find (19).

We note in passing that expansion (37) may be compared with one given by Bromwich [3, p. 199, ex. 17] which is

$$\text{Arctan } x = \sum_{n=0}^{\infty} (-1)^n \binom{3n+1}{n} \frac{t^{2n+1}}{2n+1}, \qquad t = \frac{x}{1+x^2}, \tag{38}$$

which converges, incidentally, for $|t|^2 < 4/27$. Both (37) and (38) are examples of special cases of formulas related to the Lagrange inversion formula.

We now turn to some of the older literature. Cauchy numbers have been defined by the following:

$$N_{-p,\ell,m} = \text{Constant term in expansion of } x^{-p}\left(x + \frac{1}{x}\right)^{\ell}\left(x - \frac{1}{x}\right)^{m}. \tag{39}$$

When $p = \ell + m$, then $N_{-p,\ell,m} = 1$. When $\ell + m - p$ is odd or a negative integer, then $N = 0$. Moreover,

(40) $$N_{-p,\ell,m} = \sum_{k=0}^{n} (-1)^k \binom{\ell}{n-k}\binom{m}{k}, \qquad \text{when } \ell + m - p = 2n.$$

Comparing this with (10), we see that Bruckman's A_n is given by

(41) $$A_n = N_{2n+3/2,-1/2,-1}.$$

Many interesting properties have been found for the Cauchy numbers, and the reader may consult references [5], [6], [7] and [10]. In [10], Perna numbers are defined by

(42) $$A_{m,n,\ell} = \sum_{k=0}^{n} (-1)^k \binom{n}{k}\binom{m-2n}{\ell-2k} = N_{2\ell-m,m-n,n}.$$

The late Harry Bateman (1882-1946), a master of special functions (since it was said he knew the properties of over a thousand functions, and he left dozens of card files of such information) worked on manuscripts for about 25 books, living to publish only three of them. In 1961, through the kind generosity of Professor A. Erdélyi, who was then at California Institute of Technology, Bateman's three versions of his manuscript [2] toward a book on binomial coefficients were borrowed for study at West Virginia University. A microfilm of the manuscript is on file now in the West Virginia University Library. The writer has gone through this material and edited it into a single manuscript, adding a few remarks as necessary, correcting obvious mistakes, etc. It is hoped that this version can be made more readily accessible for study by other scholars. Bateman tried to unify some of the material on binomial identities using the Cauchy number definition in one case. Here he summarized many of the properties of these and the related numbers studied by Chessin, Perna, etc. Chessin gave the formula, for example, that

$$J_a(x) = \sum_{n=0}^{\infty} \frac{N_{-a,0,a+2n}}{(a+2n)!}\left(\frac{x}{2}\right)^{a+2n}$$

for the Bessel function.

Such sums of products of two binomial coefficients continue to occur in mathematics. One example is in Graver's combinatorial work [9]. He defines coefficients $P_n(a,b)$ which turn out to be such that

$$(1 - x)^b(1 + x)^{a-b} = \sum_{n=0}^{\infty} P_n(a,b)\, x^n \; . \tag{43}$$

From this, it is easy to see that Bruckman's A_n is given by

$$A_n = P_n(-3/2, -1) \; . \tag{44}$$

Interesting formulas are found by Graver. For example:

$$P_n(a,b) = \sum_{k=0}^{n} (-1)^k \binom{b}{k}\binom{a-k}{n-k} 2^k, \qquad n \le b \le a \; . \tag{45}$$

$$P_n(a,b) = \frac{b!\,(a-b)!}{n!\,(a-n)!} P_b(a,n) \; , \tag{46}$$

expressing a symmetry in b and n, and (Graver's actual definition)

$$P_n(a,b) = \sum_{k=0}^{n} (-1)^k \binom{b}{k}\binom{a-b}{n-k} \; . \tag{47}$$

The equality of (45) and (47) is not again a new result, so extensive is the vast literature around the binomial coefficient identities. An expansion of the sort studied by Graver occurs frequently in mathematics, just as the Cauchy numbers have come to attention many times. Graver's numbers relate to Cauchy's numbers by the formula

$$P_n(\ell + m, m) = N_{-p,\ell,m} \qquad \text{with} \quad \ell + m - p = 2n \; . \tag{48}$$

In the older literature, one thing was noted as conspicuously absent; any relation of the form (6) of Bruckman or a suitable extension. Looking at Bruckman's formula in the form (23) it is tempting to generalize and wonder if by chance

$$\left\{\sum_{k=0}^{n}\binom{x}{k}\right\}^2 = \binom{x}{n}\sum_{k=0}^{n}\binom{x}{k}\frac{x-n}{x-n+k} \; ,$$

but this turns out to be false. The reader is invited to try and find such a generalization.

Bruckman's formula is an example of a case in which a certain more difficult problem is solvable. The general problem we mean is this:

PROBLEM: Let

$$f(x) = \sum_{n=0}^{\infty} A_n x^n, \qquad g(x) = \sum_{n=0}^{\infty} A_n^2 x^n,$$

for an arbitrary sequence $\{A_n\}$. How are the functions f and g related? In case $A_n = F_n = n^{th}$ Fibonacci number, we not only know the solution for squares but for any power of F_n. Other examples where the function g can be given explicitly when f is known are, e.g.:

$$\sum_{n=0}^{\infty} n x^n = x/(1-x)^2, \qquad \sum_{n=0}^{\infty} n^2 x^n = x(x+1)/(1-x)^3;$$

$$\sum_{n=1}^{\infty} \frac{x^n}{n} = \log(1-x)^{-1}, \qquad \sum_{n=1}^{\infty} \frac{x^n}{n^2} = -\int_0^x \frac{\log(1-t)}{t}\,dt;$$

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x, \qquad \sum_{n=0}^{\infty} \frac{x^n}{n!^2} = J_0(2i\sqrt{x});$$

and so on. It is clear that in general there is no really simple relation between f and g, but the writer has not found any result of this type in the literature and tosses it out as a research problem.

Solution of this problem, even with restrictions, would allow us to deal effectively with large classes of difficult problems.

In closing we mention two extensions of relation (22):

$$\sum_{n=0}^{\infty} (-1)^n \frac{t^{2n+1}}{2n+1} \sum_{k=0}^{n} \binom{x}{n-k} \frac{2n+1}{2k+1} = (1-t^2)^x \operatorname{Arctan} t, \tag{49}$$

and

$$\sum_{n=0}^{\infty} \frac{t^{2n+1}}{2n+1} \sum_{k=0}^{n} \binom{x}{n-k} \frac{2n+1}{2k+1} = (1+t^2)^x \cdot \frac{1}{2} \log \frac{1+t}{1-t}. \tag{50}$$

In a later paper we will treat some further properties of such expansions.

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For the inverse mapping $P \rightarrow Z^n$ we need

$$f_0^{-1}(p_i) = \frac{(-1)^{\epsilon}(p_i - \epsilon)}{2},$$

where

$$\epsilon = \begin{cases} 0 & \text{for } p_i \text{ even,} \\ 1 & \text{for } p_i \text{ odd} \end{cases}.$$

Then

$$\begin{aligned} f_0^{-1} f_n^{-1}(p) &= f_0^{-1}(p_1, p_2, \cdots, p_n) \\ &= \left(f_0^{-1}(p_1), \cdots, f_0^{-1}(p_n)\right) . \end{aligned}$$

6. POLYNOMIAL COUNTING FUNCTIONS

It is quite easy to see from (1) that there are at least $n!$ polynomial counting functions of P^n (obtained by permuting $p_1, p_2, \cdots, p_n$). But for $n = 3$ besides these six polynomials of degree 3, there are six more polynomials of degree 4 obtained by composition of f_2 such as