GENERATING IDENTITIES FOR FIBONACCI AND LUCAS TRIPLES

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Using the generating functions of

$$\{F_{n+m}\}_{n=0}^{\infty} \quad \text{and} \quad \{L_{n+m}\}_{n=0}^{\infty} \; ,$$

where F_{n+m} denotes the $(n+m)^{th}$ Fibonacci number and L_{n+m} denotes the $(n+m)^{th}$ Lucas number, many basic identities are easily deduced. From certain of these identities and the generating functions, we obtain identities for the triples $F_p F_q F_r$, $F_p F_q L_r$, $F_p L_q L_r$, and $L_p L_q L_r$, where p, q, and r are fixed integers.

To derive the desired generating functions we recall that

$$(0) \qquad F_{n+m} = \frac{\alpha^{n+m} - \beta^{n+m}}{\alpha - \beta} \quad \text{and} \quad L_{n+m} = \alpha^{n+m} + \beta^{n+m}$$

where

$$\alpha = \frac{1 - \sqrt{5}}{2} \quad \text{and} \quad \beta = \frac{1 + \sqrt{5}}{2} \; .$$

Note that α and β are the roots of the equation $x^2 - x - 1 = 0$, and hence $\alpha + \beta = 1$ and $\alpha\beta = -1$. The generating functions of

$$\{F_{n+m}\}_{n=0}^{\infty}$$

where m is any fixed integer is found using the given definition of F_{n+m}. We have

$$\sum_{n=0}^{\infty} F_{n+m} x^n = \sum_{n=0}^{\infty} \frac{\alpha^{n+m} - \beta^{n+m}}{\alpha - \beta} x^n$$

$$= \frac{1}{\alpha - \beta} \left[\alpha^m \sum_{n=0}^{\infty} \alpha^n x^n - \beta^m \sum_{n=0}^{\infty} \beta^n x^n \right]$$

$$(1) \qquad = \frac{1}{\alpha - \beta} \left[\alpha^m \frac{1}{1 - \alpha x} - \beta^m \frac{1}{1 - \beta x} \right]$$

$$= \frac{1}{\alpha - \beta} \left[\frac{(\alpha^m - \beta^m) - \alpha\beta(\alpha^{m-1} - \beta^{m-1})x}{(1 - \alpha x)(1 - \beta x)} \right]$$

$$= \frac{F_m + F_{m-1}x}{1 - x - x^2} \; .$$

In a similar fashion the generating function of $\{L_{n+m}\}_{n=0}^{\infty}$ is found to be

(2)
$$\sum_{n=0}^{\infty} L_{n+m} x^n = \frac{L_m + L_{m-1} x}{1 - x - x^2} .$$

(Any reader who is unfamiliar with the general theory of generating functions will find references [1], [2], [3], and [4] enlightening.)

Before considering important special cases of the above results, two lemmas are given which are proved by appropriate substitution of formulas (0).

Lemma 1. $F_n L_n = F_{2n}$, $n \in Z$, the set of integers.

Lemma 2. $F_n L_{n-1} + F_{n-1} L_n = 2F_{2n-1}$, $n \in Z$.

In utilizing formulas (1) and (2) to generate basic identities, we must first evaluate the formulas at specific values of m. It is sufficient for our purposes to consider the cases m = -2, -1, 0, 1, 2, 3, 4.

<div align="center">SPECIAL CASES OF FORMULAS (1) AND (2)</div>

(Let $1 - x - x^2 = \Delta$.)

$$\sum_{n=0}^{\infty} F_{n-2} x^n = \frac{F_{-2} + F_{-1} x}{\Delta} = \frac{-1 + 2x}{\Delta} , \qquad \sum_{n=0}^{\infty} L_{n-2} x^n = \frac{L_{-2} + L_{-3} x}{\Delta} = \frac{3 - 4x}{\Delta}$$

$$\sum_{n=0}^{\infty} F_{n-1} x^n = \frac{F_{-1} + F_{-2} x}{\Delta} = \frac{1 - x}{\Delta} , \qquad \sum_{n=0}^{\infty} L_{n-1} x^n = \frac{L_{-1} + L_{-2} x}{\Delta} = \frac{-1 + 3x}{\Delta}$$

$$\sum_{n=0}^{\infty} F_n x^n = \frac{F_0 + F_{-1} x}{\Delta} = \frac{0 + x}{\Delta} , \qquad \sum_{n=0}^{\infty} L_n x^n = \frac{L_0 + L_{-1} x}{\Delta} = \frac{2 - x}{\Delta}$$

$$\sum_{n=0}^{\infty} F_{n+1} x^n = \frac{F_1 + F_0 x}{\Delta} = \frac{1 + 0x}{\Delta} , \qquad \sum_{n=0}^{\infty} L_{n+1} x^n = \frac{L_1 + L_0 x}{\Delta} = \frac{1 + 2x}{\Delta}$$

$$\sum_{n=0}^{\infty} F_{n+2} x^n = \frac{F_2 + F_1 x}{\Delta} = \frac{1 + x}{\Delta} , \qquad \sum_{n=0}^{\infty} L_{n+2} x^n = \frac{L_2 + L_1 x}{\Delta} = \frac{3 + x}{\Delta}$$

$$\sum_{n=0}^{\infty} F_{n+3} x^n = \frac{F_3 + F_2 x}{\Delta} = \frac{2 + x}{\Delta} , \qquad \sum_{n=0}^{\infty} L_{n+3} x^n = \frac{L_3 + L_2 x}{\Delta} = \frac{4 + 3x}{\Delta}$$

$$\sum_{n=0}^{\infty} F_{n+4} x^n = \frac{F_4 + F_3 x}{\Delta} = \frac{3 + 2x}{\Delta}, \qquad \sum_{n=0}^{\infty} L_{n+4} x^n = \frac{L_4 + L_3 x}{\Delta} = \frac{7 + 4x}{\Delta} \qquad .$$

Using the fact that two series are equal if and only if the corresponding coefficients are equal, we now find several elementary identities.

Since

$$\frac{2 - x}{\Delta} = \frac{1}{\Delta} + \frac{1 - x}{\Delta} ,$$

it follows that

$$\sum_{n=0}^{\infty} L_n x^n = \sum_{n=0}^{\infty} F_{n+1} x^n + \sum_{n=0}^{\infty} F_{n-1} x^n$$

$$= \sum_{n=0}^{\infty} (F_{n+1} + F_{n-1}) x^n$$

and hence

__Lemma 3.__ $L_n = F_{n+1} + F_{n-1}$, $n \in Z^{+} \cup \{0\}$, the set of nonnegative integers.

Note from definition (0) that

$$F_{-n} = \frac{\alpha^{-n} - \beta^{-n}}{\alpha - \beta} = \frac{1}{\alpha - \beta} \left(\frac{1}{\alpha^n} - \frac{1}{\beta^n} \right)$$

(0')
$$= \frac{1}{\alpha - \beta} \frac{\beta^n - \alpha^n}{(\alpha\beta)^n} = \frac{1}{\alpha - \beta} \frac{\beta^n - \alpha^n}{(-1)^n}$$

$$= (-1)^{n+1} \frac{\alpha^n - \beta^n}{\alpha - \beta} = (-1)^{n+1} F_n$$

and

(0'')
$$L_{-n} = \alpha^{-n} + \beta^{-n} = (\alpha\beta)^{-n}(\alpha^n + \beta^n)$$

$$= (-1)^{-n} L_n = (-1)^n L_n$$

for any positive integer n.

Returning to Lemma 3, we now observe from this lemma and "definitions" (0') and (0'') that

$$F_{(-n)+1} + F_{(-n)-1} = F_{-(n-1)} + F_{-(n+1)}$$

$$= (-1)^{(n-1)+1} F_{n-1} + (-1)^{(n+1)+1} F_{n+1}$$

$$= (-1)^n [F_{n-1} + F_{n+1}]$$

$$= (-1)^n L_n = L_{-n} \qquad .$$

Hence Lemma 3 holds for <u>all</u> integers n.

In a similar manner the additional lemmas are found.

<u>Lemma 4.</u> $5F_n = L_{n+1} + L_{n-1}$, $n \in Z$.

<u>Lemma 5.</u> $2F_{n+1} = F_n + L_n = F_{n+2} + F_{n-1}$, $n \in Z$.

<u>Lemma 6.</u> $2F_{n-1} = F_n - L_n$, $n \in Z$.

<u>Lemma 7.</u> $F_{n+3} = F_{n+1} + L_{n-1}$, $n \in Z$.

<u>Lemma 8.</u> $3F_n = L_{n+1} - F_{n-1}$, $n \in Z$.

<u>Lemma 9.</u> $3L_{n+2} = L_n + L_{n+4}$, $n \in Z$.

<u>Lemma 10.</u> $3F_{n+2} = F_n + F_{n+4}$, $n \in Z$.

<u>Lemma 11.</u> $2F_{n+1} = L_{n+1} - F_{n-2}$, $n \in Z$.

<u>Lemma 12.</u> $L_n + F_n = F_{n+2} + F_{n-1}$, $n \in Z$.

Although these results are of interest in themselves, their principal use is as lemmas to more profound results. The reader is encouraged to consider additional special cases of formulas (0), and then generate additional Fibonacci and Lucas identities.

The next three results are also generated from formulas (1) and (2). These fundamental identities are essential to our development of Fibonacci and Lucas triples.

<u>Theorem 1.</u> $F_n L_m + F_{n-1} L_{m-1} = L_{n+m-1}$, for any $n,m \in Z$.

<u>Proof.</u> Let m be any fixed integer. Then

$$\sum_{n=0}^{\infty} (F_n L_m + F_{n-1} L_{m-1})x^n = L_m \sum_{n=0}^{\infty} F_n x^n + L_{m-1} \sum_{n=0}^{\infty} F_{n-1} x^n$$

$$= L_m \frac{x}{\Delta} + L_{m-1} \frac{(1-x)}{\Delta}$$

$$= \frac{L_{m-1} + (L_m - L_{m-1})x}{\Delta}$$

$$= \frac{L_{m-1} + L_{m-2}x}{\Delta}$$

$$= \sum_{n=0}^{\infty} L_{n+m-1} x^n \qquad ,$$

by formula (2). Results (0') and (0'') complete the proof.

From a development similar to the above proof, we find a companion result to Theorem 1.

<u>Theorem 2.</u> $F_n F_m + F_{n-1} F_{m-1} = F_{n+m-1}$, for any $n,m \in Z$.

<u>Theorem 3.</u> $L_n L_m + L_{n-1} L_{m-1} = L_{n+m} + L_{n+m-2} = 5F_{n+m-1}$, for any $n,m \in Z$.

<u>Proof.</u> Since $L_{n+m} + L_{n+m-2} = 5F_{n+m-1}$ by Lemma 4, we need only consider the first part of the identity. Let m be any fixed integer. Now

$$\sum_{n=0}^{\infty} (L_n L_m + L_{n-1} L_{m-1}) x^n = L_m \sum_{n=0}^{\infty} L_n x^n + L_{m-1} \sum_{n=0}^{\infty} L_{n-1} x^n$$

$$= L_m \left(\frac{2 - x}{\Delta} \right) + L_{m-1} \left(\frac{-1 + 3x}{\Delta} \right)$$

$$= \frac{[L_m + (L_m - L_{m-1})] + [2L_{m-1} + (L_{m-1} - L_m)]x}{\Delta}$$

$$= \frac{[L_m + L_{m-2}] + [L_{m-1} + (L_{m-1} - L_{m-2})]x}{\Delta}$$

$$= \frac{L_m + L_{m-1}x}{\Delta} + \frac{L_{m-2} + L_{m-3}x}{\Delta}$$

$$= \sum_{n=0}^{\infty} (L_{n+m} + L_{n+m-2}) x^n \qquad .$$

Aided by the partial fractions technique we find the final result needed to generate the specified Fibonacci and Lucas triples. It is the following:

$$\frac{(p + qx)}{\Delta} \frac{(r + tx)}{\Delta} = \frac{pr + (pt + qr)x + qtx^2}{\Delta^2}$$

(3)

$$= \frac{-qt}{\Delta} + \frac{(pr + qt) + (pt + qr - qt)x}{\Delta^2} \qquad .$$

The identities are now found by convoluting series (generating functions) of the forms (1) and (2). We begin by specifying m and s as fixed integers. Now

$$\frac{F_m + F_{m-1}x}{\Delta} \cdot \frac{L_s + L_{s-1}x}{\Delta} = \sum_{n=0}^{\infty} F_{n+m} x^n \sum_{n=0}^{\infty} L_{n+s} x^n$$

(4)

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{n} F_{k+m} L_{n-k+s} x^n \qquad ,$$

and by Eq. (3) this product also equals

$$\frac{-F_{m-1}L_{s-1}}{\Delta} + \frac{(F_m L_s + F_{m-1} L_{s-1}) + (F_m L_{s-1} + F_{m-1} L_s - F_{m-1} L_{s-1})x}{\Delta^2}$$

$$= \frac{-F_{m-1}L_{s-1}}{\Delta} + \frac{L_{m+s-1} + (F_{m-1} L_s + F_{m-2} L_{s-1})x}{\Delta^2}$$

by Theorem 1 and substitution of F_{m-2} for $F_m - F_{m-1}$.

$$= -F_{m-1}L_{s-1} \frac{1}{\Delta} + \frac{L_{m+s-1} + L_{m+s-2}}{\Delta} x \frac{1}{\Delta}$$

by Theorem 1

$$= -F_{m-1}L_{s-1} \sum_{n=0}^{\infty} F_{n+1} x^n + \sum_{n=0}^{\infty} L_{n+m+s-1} x^n \cdot \sum_{n=0}^{\infty} F_{n+1} x^n$$

by definition of generating functions (1) and (2)

$$= \sum_{n=0}^{\infty} [-F_{m-1}L_{s-1}F_{n+1}] x^n + \sum_{n=0}^{\infty} \sum_{k=0}^{n} F_{k+1}L_{n-k+m+s-1} x^n$$

(5)
$$= \sum_{n=0}^{\infty} \left[-F_{m-1}L_{s-1}F_{n+1} + \sum_{k=0}^{n} F_{k+1}L_{n-k+m+s-1}\right] x^n .$$

By equating the coefficients of series (4) and (5), the first identity is deduced. It may be expressed as

$$\sum_{k=0}^{n} F_{k+m}L_{n-k+s} = -F_{m-1}L_{s-1}F_{n+1} + \sum_{k=0}^{n} F_{k+1}L_{n-k+m+s-1}$$

or

$$F_{m-1}L_{s-1}F_{n+1} = \sum_{k=0}^{n} (F_{k+1}L_{n-k+m+s-1} + F_{k+m}L_{n-k+s}) .$$

Letting $p = m - 1$, $q = n + 1$, and $r = s - 1$, the identity becomes

Theorem 4.

$$F_p F_q L_r = \sum_{k=0}^{q-1} (F_{k+1}L_{p+q+r-k-1} + F_{p+k+1}L_{q+r-k}) ,$$

for any integers p, q, and r.

One notes the need of definitions (0') and (0'') if any of the above integers is negative.

Following the procedure given above, aided by the given lemmas, Theorems 1-3, and definitions, two additional identities are found. The first is a result of the convolution of

$$\frac{F_m + F_{m-1}x}{\Delta}$$

with

$$\frac{F_t + F_{t-1}x}{\Delta} \ ,$$

and the second is determined by the convolution of

$$\frac{L_m + L_{m-1}x}{\Delta}$$

with

$$\frac{L_t + L_{t-1}x}{\Delta} \ .$$

Theorem 5.

$$F_p F_q F_r = \sum_{k=0}^{r-1} (F_{p+q+k+1}F_{r-k} - F_{p+k+1}F_{r+q-k}) \ ,$$

for any $p, q, r \in Z$.

Theorem 6.

$$F_p L_q L_r = \sum_{k=0}^{p-1} (5F_{p-k}F_{q+r+k+1} - L_{q+k+1}L_{p+r-k}) \ ,$$

for any $p, q, r \in Z$.

Theorem 7.

$$L_p L_q L_r = 5 \left[\sum_{k=0}^{p-2} (F_{q+r+k+1}L_{p-k} - F_{p+r-k}L_{q+k+1}) - F_{p+q+r} \right] - L_{p+q}L_{r+1} \ ,$$

for any $p, q, r \in Z$.

Proof. From Lemma 3, we obtain

$$L_p L_q L_r = (F_{p+1} + F_{p-1})L_q L_r$$

$$= F_{p+1}L_q L_r + F_{p-1}L_q L_r \ .$$

Now from Theorem 6, it follows that

$$L_p L_q L_r = \sum_{k=0}^{p} (5F_{p-k+1}F_{q+r+k+1} - L_{q+k+1}L_{p+r-k+1})$$

$$+ \sum_{k=0}^{p-2} (5F_{p-k-1}F_{q+r+k+1} - L_{q+k+1}L_{p+r-k-1})$$

$$= \sum_{k=0}^{p-2} [5F_{q+r+k+1}(F_{p-k+1} + F_{p-k-1}) - L_{q+k+1}(L_{p+r-k+1} + L_{p+r-k-1})]$$

$$+ (5F_2 F_{p+q+r} - L_{p+q}L_{r+2}) + (5F_1 F_{p+q+r+1} - L_{p+q+1}L_{r+1})$$

$$= 5\sum_{k=0}^{p-2} (F_{q+r+k+1}L_{p-k} - F_{p+r+k}L_{q+k+1})$$

$$+ 5F_{p+q+r+2} - (5F_{p+q+r+1} + L_{p+q}L_{r+1})$$

by Lemmas 2 and 4 and Theorem 4

$$= 5\left[\sum_{k=0}^{p-2} (F_{q+r+k+1}L_{p-k} - F_{p+r-k}L_{q+k+1}) - F_{p+q+r}\right] - L_{p+q}L_{r+1} \quad .$$

Many corollaries to the last three theorems are immediate by making substitution(s) for p, q, and r, respectively, in the given identities. The formulation and derivation of these results we leave to the reader.

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