AN OLD FIBONACCI FORMULA AND STOPPING RULES

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A fair coin is tossed, a head giving a return of +1, a tail of -1. Let the sum of these returns for a sequence of m throws be designated S_m. We define a stopping rule for the sequence: The sequence of throws will end if S_m is outside the closed interval -2 to +1.

At the end of m throws, if all possible variations are considered, there will be a certain number of 1's, 0's, -1's and -2's which will be designated n(1), n(0), n(-1), and n(-2), respectively. The number of sequences that terminate at m because of the stopping rules will be denoted $\phi(m)$.

Let us consider the first few steps. At the end of the first throw, there are two possible values +1, -1, and no terminations. Hence n(1) = 1, n(-1) = 1, $\phi(1) = 0$.

At the end of two throws, the possible values are +2, 0, 0, -2, the first being a termination. Hence n(0) = 2, n(-2) = 1, $\phi(2) = 1$. Continuing with the non-terminating sequences, we have values -1, +1, -1, +1, -3, -1 at the end of three throws. Hence $\phi(3) = 1$, n(1) = 2, n(-1) = 3.

The following table summarizes a few additional steps.

	m								
	1	2	3	4	5	6	2m - 1	2m	2m + 1
$\phi(m)$	0	1	1	2	3	5	F_{2m-2}	F_{2m-1}	F_{2m}
n(1)	1	0	2	0	5	0	F_{2m-1}	0	F_{2m+1}
n(0)	0	2	0	5	0	13	0	F_{2m+1}	0
n(-1)	1	0	3	0	8	0	F_{2m}	0	F_{2m+2}
n(-2)	0	1	0	3	0	8	0	F_{2m}	0

The general pattern is shown under the columns 2m - 1, 2m, 2m + 1. Now assume that we have the pattern in column 2m - 1. The 1's get out of bounds at 2 giving $\phi(2m) = F_{2m-1}$. The 1's and -1's combine to give $F_{2m-1} + F_{2m} = F_{2m+1}$ zeros. The -1's go to -2 giving F_{2m}. Starting at 2m, the -2's go out of bounds giving $\phi(2m+1) = F_{2m}$. The 0's and -2's combine to give $F_{2m} + F_{2m+1} = F_{2m+2}$ for -1. The 0's also go to 1 putting F_{2m+1} in that place. Thus the process is seen to continue indefinitely.

REFERENCE

A. Wald, "On Cumulative Sums of Random Variables," Annals of Mathematical Statistics, Vol. 15 (1944), p. 281.