LINEAR DIFFERENCE EQUATIONS AND GENERALIZED CONTINUANTS
PART I: ALGEBRAIC DEVELOPMENTS

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1. INTRODUCTION

A continuant determinant (or matrix) has elements in the diagonals through (1,1), (1,2), and (2,1) only, and zeros elsewhere. We can use the notation $K_s(h_1{}^{g_1}_{g_1'})$ for the s^{th} order continuant, where

(1) $$\begin{vmatrix} h_1 & g_1 & 0 & & \\ g_1' & h_2 & g_2 & & \\ 0 & g_2' & & & \\ & & & & g_s \\ & & & g_s' & h_s \end{vmatrix}_{(s)}$$

As is well known, by expanding this by its last row and column, we find the recurrence relation (omitting the arguments for brevity)

(2) $$K_s = h_s K_{s-1} - g_s' g_s K_{s-2} \qquad s = 2, 3, \cdots$$

with $K_0 = 1$, $K_1 = h_1$. Note that K_s is unchanged in value if the signs are changed for any subset of the g's along with the corresponding subset of the g''s. Again note that the usual Fibonacci sequence arises from either $g_\lambda = 1$, $g_\lambda' = -1$ (or of course $g_\lambda = -1$, $g_\lambda' = 1$) or $g_\lambda = g_\lambda' = i$ $(= \sqrt{-1})$.

Many elementary properties of recursive schemes such as (2) are well known and in particular Brother Alfred Brousseau [1] has given some of these in the case when the coefficients are constants.

The question arises as to what happens when we add diagonals to (1) through (1,3) and (3,1) and produce a 5-diagonal determinant. We shall call a $(2s + 1)$ diagonal determinant (with elements in the main diagonal and the s super-diagonals, and the s sub-diagonals) a continuant of degree s. The recursions followed by these generalized continuants have been studied by H. D. Ursell [2]. In fact, Ursell gives the following table which refers to the order of the difference equation satisfied by a continuant of degree s:

Order of Recurrence Relation

Degree s	1	2	3	4	5	6
Symmetric Case	2	5	15	49	169	604
Unsymmetric Case	2	6	20	70	252	924

The rate of increase of the difference equation order is very remarkable.

2. THE FIVE DIAGONAL SYMMETRIC CONTINUANT

We use the notation $K_s(h_1, g_1, f_1)$ for a second-degree symmetric continuant with elements $h_1, h_2, \cdots,$ in the principal diagonal, $g_1, g_2, \cdots,$ on the diagonal through (1, 2) and (2, 1), $f_1, f_2, \cdots,$ on the diagonals through (1, 3) and (3, 1) and zeros elsewhere. The fifth-order recurrence is then given by (see [3], p. 173, expression (16))

$$(3) \qquad g_{s-2}K_s = a_sK_{s-1} - b_s(g_{s-1}K_{s-2} - g_{s-2}f_{s-2}K_{s-3}) - f^2_{s-3}f_{s-2}c_sK_{s-4} + f_{s-2}f^2_{s-3}f^2_{s-4}g_{s-1}K_{s-5}$$

where $s = 3, 4, \cdots,$ with

$$K_{-2} = K_{-1} = 0, \quad K_0 = 1, \quad K_1 = h_1 ,$$
$$K_2 = h_1h_2 - g_1^2 ,$$

where

$$a_s = h_sg_{s-2} - f_{s-2}g_{s-1} ,$$
$$b_s = g_{s-1}g_{s-2} - h_{s-1}f_{s-2} ,$$
$$c_s = h_{s-2}g_{s-1} - f_{s-2}g_{s-2} .$$

We discuss several special cases.

2.1 $g_1 = g_2 = \cdots = g_{s-1} = 0$. We now have to expand K_s by its last row and column since formula (3) aborts. We find

$$(4) \qquad K_s = h_sK_{s-1} - f^2_{s-2}h_{s-1}K_{s-3} + f^2_{s-2}f^2_{s-3}K_{s-4} \qquad (s = 4, 5, \cdots)$$

with

$$K_0 = 1 ,$$
$$K_1 = h_1 ,$$
$$K_2 = h_1h_2 ,$$
$$K_3 = h_2(h_1h_3 - f_1^2) .$$

Using (4) we find for the next few cases,

$$K_4 = (h_1h_3 - f_1^2)(h_2h_4 - f_2^2) ,$$
$$K_5 = (h_2h_4 - f_2^2)\left(h_5(h_1h_3 - f_1^2) - h_1f_3^2\right)$$

indicating that K_s is the product of two continuants of degree 1 (three diagonals). This is easily seen from the determinant for K_s by expanding by sub-matrices consisting of elements from odd rows (and columns). For example,

$$(5)\qquad K_7 = \begin{vmatrix} h_1 & f_1 & 0 & 0 \\ f_1 & h_3 & f_3 & 0 \\ 0 & f_3 & h_5 & f_5 \\ 0 & 0 & f_5 & h_7 \end{vmatrix} \begin{vmatrix} h_2 & f_2 & 0 \\ f_2 & h_4 & f_4 \\ 0 & f_4 & h_6 \end{vmatrix}$$

and this type of condensation has been given by Muir [4]. We may verify directly from (4) that K_s does in fact factor, and defining first degree continuants

$$(6a)\qquad K_s^{(2)}(h_1, f_1) = \begin{vmatrix} h_1 & f_1 & & \\ f_1 & h_3 & & \\ & & & f_{2s-3} \\ & & f_{2s-3} & h_{2s-1} \end{vmatrix}_{(s)}$$

$$(6b)\qquad K_s^{(2)}(h_2, f_2) = \begin{vmatrix} h_2 & f_2 & & \\ f_2 & h_4 & & \\ & & & f_{2s-2} \\ & & f_{2s-2} & h_{2s} \end{vmatrix}_{(s)}$$

it can be demonstrated that

$$(7)\qquad \begin{aligned} K_{2s}(h_1, 0, f_1) &= K_s^{(2)}(h_1, f_1)K_s^{(2)}(h_2, f_2)\ , \\ K_{2s+1}(h_1, 0, f_1) &= K_{s+1}^{(2)}(h_1, f_1)K_s^{(2)}(h_2, f_2)\ . \end{aligned}$$

In particular taking $h_s = 1$, $f_s = i$ in (4) we see that the sequence (K_s) where

$$(8)\qquad K_s = K_{s-1} + K_{s-3} + K_{s-4} \qquad (s = 4, 5, \cdots)$$

with $K_0 = 1$, $K_1 = 1$, $K_2 = 1$, $K_3 = 2$, is such that K_{2s-1} is the product of consecutive Fibonacci numbers whereas K_{2s} is the square of a Fibonacci number. For example,

s	4	5	6	7	8	9	10	11	12
K_s	2^2	$2\cdot3$	3^2	$3\cdot5$	5^2	$5\cdot8$	8^2	$8\cdot13$	13^2

It is perhaps not surprising to find the characteristic equation of (8) has zeros $\pm i$, $(1 \pm \sqrt{5})/2$, and indeed

$$(9)\qquad K_s = \frac{(2-i)}{10}\, i^s + \frac{(2+i)}{10}\, (-i)^s + \left(\left(\frac{1+\sqrt{5}}{2}\right)^{s+2} + \left(\frac{1-\sqrt{5}}{2}\right)^{s+2}\right)\Big/ 5\ .$$

Again since the characteristic equation has a zero with largest modulus, then

$$\lim_{s\to\infty} \frac{K_{s+1}}{K_s} = \frac{1+\sqrt{5}}{2}\ .$$

2.2 Constant Elements in the Diagonals.

We consider K_s (h, g, f) where h, g, f are either unity in modulus, or zero. The following seem to be the most interesting:

Case	h	g	f
1	0	1	1
2	1	1	-1
3	1	i	-1
4	1	i	1

$(i = \sqrt{-1})$

Case 1

$$K_s = -K_{s-1} - K_{s-2} + K_{s-3} + K_{s-4} + K_{s-5} \qquad s = 3, 4, \cdots$$

with

$$K_{-2} = K_{-1} = 0, \quad K_0 = 1, \quad K_1 = 0, \quad K_2 = -1 .$$

In addition

s	3	4	5	6	7	8	9	10	11	12
K_s	2	0	-2	3	0	-3	4	0	-4	-3

Characteristic Equation

$$(x - 1)(x^2 + x + 1)^2 = 0$$

Roots $\quad x = 1, w, w, w^2, w^2$, where w is a primitive cube root of unity.

Explicit Formula

$$K_s = \frac{2}{9} + \left(1 + 4w + s(1 + 2w)\right)\left(\frac{w}{9}\right)^{s-1} - \left(1 - 3w + s(1 - w)\right)\left(\frac{w}{9}\right)^{2s-1}$$

from which

$$K_{3s} = s + 1, \quad K_{3s+1} = 0, \quad K_{3s+2} = -s - 1 .$$

Case 2

$$K_s = 2K_{s-1} - 2K_{s-2} - 2K_{s-3} + 2K_{s-4} - K_{s-5}$$

s	K_s	Δ_s $\left(= \sqrt{(K_s^2 - K_{s-1}K_{s+1})}\right)$
0	1	—
1	1	1
2	0	2
3	-4	4
4	-8	6
5	-7	11
6	9	19
7	40	32
8	64	56
9	24	96
10	-135	165
11	-375	285
12	-440	490
13	124	844
14	1584	1454
15	3185	2503

Characteristic Roots

$$x_1 = \left(\sqrt{3}\, e^{i\pi/6} + \sqrt[4]{13}\, e^{i\alpha/2}\right)/2\,,$$

$$x_2 = 1/x_1\,,$$

$$x_3 = \bar{x}_1 \quad \text{(conjugate)}\,,$$

$$x_4 = 1/\bar{x}_1\,,$$

$$x_5 = 1\,,$$

where $\tan\alpha = 3\sqrt{3}/5$.

The roots of greatest modulus being complex, "explains" the apparently unpredictable behavior of K_s. On the other hand, notice that $K_s^2 - K_{s-1}K_{s+1}$ is always a perfect square, and in fact Δ_s follows the recurrence

$$\Delta_s = \Delta_{s-1} + \Delta_{s-2} + \Delta_{s-3} - \Delta_{s-4} \qquad (s = 2, \cdots)$$

with

$$\Delta_{-1} = 0, \qquad \Delta_0 = 1, \qquad \Delta_1 = 1\,,$$

and characteristic roots

$$x_1 = -\left(\sqrt{13} + 1 + \sqrt{(2\sqrt{13} - 2)}\right)/4\,,$$

$$x_2 = -\left(\sqrt{13} + 1 - \sqrt{(2\sqrt{13} - 2)}\right)/4\,,$$

$$x_3 = \left(\sqrt{13} - 1 + i\sqrt{(2\sqrt{13} + 2)}\right)/4\,,$$

$$x_4 = \left(\sqrt{13} - 1 - i\sqrt{(2\sqrt{13} + 2)}\right)/4\,,$$

in which x_1 has the greatest numerical value, and $|x_3| = |x_4| = 1$. Actually it can be shown that

$$\lim_{s\to\infty} \frac{\Delta_{s+1}}{\Delta_s} = \frac{\sqrt{13} + 1 + \sqrt{2(\sqrt{13} - 1)}}{4}\,.$$

Case 3

$$K_s = 2K_{s-1} + 2K_{s-4} - K_{s-5} \qquad (s = 4, 5, \cdots)$$

with

s	0	1	2	3	4	5	6	7	8	9	10
K_s	1	1	2	4	10	21	45	96	208	432	933

Characteristic Roots

$$x_1 = 1$$

$$x_{2,3,4,5} = \frac{3 \pm \sqrt{5} \pm \sqrt{(6\sqrt{5} - 2)}}{4}$$

$$\text{Magnitude of largest root} = \left(3 + \sqrt{5} + \sqrt{(6\sqrt{5} - 2)}\right)/4$$

$$\lim_{s\to\infty} \frac{K_{s+1}}{K_s} = \frac{3 + \sqrt{5} + \sqrt{(6\sqrt{5} - 2)}}{4}$$

$$= 2.1537$$

Comments (i) K_s is always positive

(ii) $\sqrt{|K_{s+1}K_{s-1} - K_s^2|}$ is an integer.

Case 4

$$K_s = 2K_{s-2} - 2K_{s-3} + K_{s-5}$$

s	0	1	2	3	4	5	6	7	8	9	10
K_s	1	1	2	0	2	-3	5	-8	16	-24	45

Characteristic Roots

$$x_1 = 1 \qquad x_{2,3} = \frac{-(1+\sqrt{13}) \pm \sqrt{2\sqrt{13} - 2}}{4}$$

$$x_{4,5} = \frac{-(1-\sqrt{13}) \pm i\sqrt{2\sqrt{13} + 2}}{4}$$

$$\lim_{s\to\infty} \frac{K_{s+1}}{K_0} = -\frac{1}{4}\left\{(1+\sqrt{13}) + \sqrt{2\sqrt{13} - 2}\right\}$$

$$= -1.7221 \qquad .$$

3. FACTORABLE CONTINUANTS

A number of these have been given by D. E. Rutherford [5], [6]. In particular, Rutherford remarks that the n^{th} Fibonacci number can be expressed as

$$\prod_{r=1}^{n-1}\left(1 - 2i\cos\frac{r\pi}{n}\right) . \tag{10}$$

Moreover, although he does not give the recurrence relation, he quotes the factors of (in our notation) $K_s\ (z, 2a, 1)$, where

$$K_s \equiv \begin{vmatrix} z, & 2a, & 1, & & & & & \\ 2a, & z, & 2a, & 1, & & & & \\ 1, & 2a, & z, & 2a, & 1, & & & \\ & 1, & 2a, & z, & 2a, & 1, & & \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ & & & & & 1, & 2a, & z \end{vmatrix}_{(s)}$$

as

$$\frac{1}{2(\cos 2\alpha - \cos 2\beta)}\left\{\frac{\sin^2(s+2)\alpha}{\sin^2\alpha} - \frac{\sin^2(s+2)\beta}{\sin^2\beta}\right\} ,$$

where

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