A TRIANGLE WITH INTEGRAL SIDES AND AREA

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The object of this paper is to discuss the problem [3] of finding all triangles having integral area and consecutive integral sides. The class of all such triangles is determined uniquely by a simple recurrent sequence. We also examine other interesting sequences associated with the triangles. Such triangles have been of interest since the time of Heron of Alexandria and the reader is referred to Dickson's monumental history [9, Vol. 2, Chapter 5] for a detailed account of this and similar problems up to 1920.

The area, K, of a triangle having sides a, b, c must satisfy the formula of Heron

$$K^2 = s(s - a)(s - b)(s - c) ,$$

where

$$s = (a + b + c)/2 .$$

Letting the sides of our triangle be $u - 1$, u, $u + 1$, we have $s = 3u/2$ and the equation

$$K^2 = \frac{3u^2(u^2 - 4)}{16} . \tag{1}$$

Evidently u must be even; for if u were odd then both u^2 and $u^2 - 4$ would be odd and 16 could not divide into the numerator. In order for 3N to be a perfect square it is necessary that N be a multiple of 3. However, u^2 cannot be a multiple of 3 without also being a multiple of 9, and so the only way to account for the factor 3 in the numerator is to impose the Diophantine equation $u^2 - 4 = 3v^2$, or

$$u^2 - 3v^2 = 4 . \tag{2}$$

All solutions to the problem will be determined by solving this equation for u, making certain that we obtain even values of u.

Equation (2) is of the general class $u^2 - Dv^2 = 4$ and a complete solution of this equation may be found in LeVeque [5, Vol. 1, p. 145]. The substance of the solution, as it applies to our work is that if $u_1 + v_1\sqrt{D}$ is the minimal positive solution of $u^2 - Dv^2 = 4$, $D \neq$ square, $D > 0$, then the general solution for positive u, v is given by the symbolic formula

$$u + v\sqrt{D} = 2\left(\frac{u_1 + v_1\sqrt{D}}{2}\right)^n , \qquad (n = 0, 1, 2, \cdots)$$

where v and u are found by expanding the right-hand side by the binomial theorem and equating radical and non-radical parts. It is easily seen that the minimal positive solution of (2) is $4 + 2\sqrt{3}$ so that the general solution is given by

$$u + v\sqrt{3} = 2(2 + \sqrt{3})^n = 2\sum_{k=0}^{n}\binom{n}{k} 2^{n-k} (\sqrt{3})^k$$

$$= 2\sum_{k=0}^{[n/2]}\binom{n}{2k} 2^{n-2k}3^k + 2(\sqrt{3})\sum_{k=0}^{[(n-1)/2]}\binom{n}{2k+1} 2^{n-2k-1}3^k .$$

Thus we have

$$u = 2^{n+1}\sum_{k=0}^{[n/2]}\binom{n}{2k}(3/4)^k .$$

However, it is easy to split up the binomial expansion and obtain the well-known formula

$$\sum_{k=0}^{[n/2]}\binom{n}{2k} x^k = \frac{1}{2}\left\{(1 + \sqrt{x})^n + (1 - \sqrt{x})^n\right\} ,$$

whence we have

$$u = u_n = (2 + \sqrt{3})^n + (2 - \sqrt{3})^n, \qquad (n = 0, 1, 2, \cdots) . \tag{3}$$

It is of interest to point out that we could also write

$$u_n = \frac{(1 + \sqrt{3})^{2n} + (1 - \sqrt{3})^{2n}}{2^n} , \tag{4}$$

but the former relation is easier to use in practice. We also remark that it is easy to prove by induction that u as determined by (3) is indeed even. A shorter derivation of (3) is to note that

$$2u = (u + v\sqrt{3}) + u - v\sqrt{3}) = 2(2 + \sqrt{3})^n + 2(2 - \sqrt{3})^n .$$

Cf. the solution given by E. P. Starke [7].

We also have the recurrence relation

(5) $$u_{n+2} = 4u_{n+1} - u_n, \qquad (u_0 = 2,\ u_1 = 4)$$

since this recurrence is associated with the characteristic equation

$$x^2 = 4x - 1$$

whose roots are $2 + \sqrt{3}$, $2 - \sqrt{3}$. The recurrence relation allows us to compute a short table of values of u, as follows:

n	$u = u_n$
0	2
1	4
2	14
3	52
4	194
5	724
6	2702
7	10084
8	37634
9	140452
10	524174
11	1956244
12	7300802
13	27246964
14	101687054
15	379501252
16	1416317954
17	5285770564
18	19726764302
19	73621286644
20	275758382274

Actually our problem is an old one, rational triangles having always been of interest. A solution of the form (3) was given, for example, by Reinhold Hoppe in 1880 [4]. Also, Cf. solutions in [7], [8].

The first six triangles, together with their areas, are:

1,	2,	3,	0
3,	4,	5,	6
13,	14,	15,	84
51,	52,	53,	1170
193,	194,	195,	16296
723,	724,	725,	228144

The triangle 3, 4, 5 is the only right triangle in the sequence because $(u - 1)^2 + u^2 = (u + 1)^2$ implies $u(u - 4) = 0$ which has only the one non-trivial solution. The triangle 13, 14, 15 has been used widely in the teaching of geometry. In fact the writer first became aware of this example during a course in college where the triangle was used as a standard reference triangle. Such a triangle has rational values for its major constants, as we shall see here, and so makes it possible to have problems with 'nice' answers. For example, in this case the sines of the three angles in the triangle are 4/5, 12/13, and 56/65. The radii of the escribed circle are 21/2, 14, and 12. The altitudes are 168/13, 12, and 168/15. Cf. [7].

It is easy to conjecture that the area $K = K_n$ satisfies the recurrence relation

$$K_{n+2} = 14K_{n+1} - K_n, \qquad (K_0 = 0, \ K_1 = 6) . \tag{6}$$

If this were true, we could find an explicit formula for K since the characteristic equation for (6) is $x^2 - 14x + 1 = 0$, whose roots are $7 \pm 4\sqrt{3}$. For suitable constants A, B we should then have

$$K_n = A(7 + 4\sqrt{3})^n + B(7 - 4\sqrt{3})^n.$$

From the initial values, A, B are easily determined and we find that

$$K_n = \frac{\sqrt{3}}{4} \left\{(7 + 4\sqrt{3})^n - (7 - 4\sqrt{3})^n\right\}$$

which simplifies to

$$K_n = \frac{\sqrt{3}}{4} \left\{(2 + \sqrt{3})^{2n} - (2 - \sqrt{3})^{2n}\right\} . \tag{7}$$

According to the review in the Fortschritte [4] it was in this form that Hoppe found the area.

Now (7) follows from (6) which we conjectured from tabular values of K. However it is easy to show that K_n given by (7) satisfies (6). Thus we shall prove (6) by proving (7) in a novel way, as follows.

By (1) we have, for any triangle T_n,

$$16K_n^2 = 3u_n^2(u_n^2 - 4) ,$$

and it is easy to see that (3) implies

$$u_n^2 = u_{2n} + 2 , \tag{8}$$

whence

$$16K_n^2 = 3(u_{2n} + 2)(u_{2n} - 2) = 3(u_{2n}^2 - 4) = 3(u_{4n} - 2) ,$$

so that we have the formula

(9) $$K_n^2 = \frac{3}{16}(u_{4n} - 2) .$$

Thus

(10) $$K_n = \frac{\sqrt{3}}{4}(u_{4n} - 2)^{\frac{1}{2}} .$$

However a short calculation shows that in fact

$$\left\{(2+\sqrt{3})^{2n} - (2-\sqrt{3})^{2n}\right\}^2 = (2+\sqrt{3})^{4n} + (2-\sqrt{3})^{4n} - 2$$
$$= u_{4n} - 2 ,$$

whence formula (10) gives (7) which we wanted to prove.

We remark that relation (8) is very useful in checking a table of u_n and was used for this purpose here to be certain of the value of u_{20}.

The radius, r, of the inscribed circle of any triangle is given by the formula $K = rs$. In the case at hand this gives

(11) $$r^2 = r_n^2 = \frac{K^2}{a^2} = \frac{u^2 - 4}{12} = \frac{u_n^2 - 4}{12} = \frac{u_{2n} - 2}{12}$$

and it is easy to prove that

(12) $$r_{n+2} = 4r_{n+1} - r_n, \qquad (r_0 = 0,\ r_1 = 1)$$

so that every triangle T_n has an integral inradius. The first few values of r are 0, 1, 4, 15, 56, 209, 780, 2911, 10864, $\cdots$.

Noting that recurrence relation (12) is the same as relation (5) we suspect that there are other intimate relations between u and r. Indeed, the theory of continued fractions provides us an interesting result. Some very handy information on continued fractions is given by Davenport [2] and especially the table on page 105. First of all, our original equation (2) may be transformed as follows. Since u is even, say $u = 2x$, we have $4x^2 - 3y^2 = 4$, whence v is even, say $v = 2y$, and so the equation can be written as

(13) $$x^2 - 3y^2 = 1$$

which suggests that we examine the familiar continued fraction expansion for $\sqrt{3}$. Indeed,

$$\sqrt{3} = 1 + \frac{1}{1+}\,\frac{1}{2+}\,\frac{1}{1+}\,\frac{1}{2+}\,\frac{1}{1+}\,\frac{1}{2+}\cdots$$

and the first few convergents are

$$\frac{1}{1}, \frac{2}{1}, \frac{5}{3}, \frac{7}{4}, \frac{19}{11}, \frac{26}{15}, \frac{71}{41}, \frac{97}{56}, \cdots .$$

The interesting point here is that every other numerator is one-half u_n, while every other denominator is precisely r_n. By means of some simple transformations we can bring out the relation more strikingly. In fact the continued fraction

$$C = 1 + \frac{1}{1+}\,\frac{1}{3-}\,\frac{1}{4-}\,\frac{1}{4-}\,\frac{1}{4-}\,\frac{1}{4-}\,\frac{1}{4-}\,\frac{1}{4-}\,\frac{1}{4-}\,\frac{1}{4-}\cdots \tag{14}$$

has successive convergents

$$\frac{1}{1}, \frac{2}{1}, \frac{7}{4}, \frac{26}{15}, \frac{97}{56}, \frac{362}{209}, \frac{1351}{780}, \cdots$$

so that each numerator is $\frac{1}{2}u$ and each denominator is r. It can be shown that the continued fraction (14) converges to $\sqrt{3}$. Let us show that $\frac{1}{2}u/r$ also tends to $\sqrt{3}$. We have, by (11)

$$\frac{1}{4}\,\frac{u^2}{r^2} = 3\,\frac{u^2}{u^2-4} = 3\,\frac{1}{1-\dfrac{4}{u^2}} \to 3 \text{ as } n \to \infty ,$$

so that we can say that our general T_n has the interesting property that

$$\lim_{n\to\infty} \frac{u_n}{r_n} = 2\sqrt{3} . \tag{15}$$

It is interesting to recall Heron's formula (iterative) for the square root of 3 :

$$a_{n+1} = \frac{5a_n + 9}{3a_n + 5} .$$

Starting with $a_1 = 5/3$ we find the successive approximations

$$\frac{5}{3}, \frac{26}{15}, \frac{265}{153}, \frac{1351}{780}, \cdots$$

These approximations, especially the value 1351/780, are of historical interest.

One may find formulas for the radii of the escribed circles for the class T_n by recalling that [1, p. 12]

$$rs = (s - a)r_a = (s - b)r_b = (s - c)r_c .$$

Further interesting relations follow from the two formulas

$$(16) \qquad r_a + r_b + r_c = r + 4R, \qquad \frac{1}{r} = \frac{1}{r_a} + \frac{1}{r_b} + \frac{1}{r_c} \,,$$

where R = radius of the circumcirle. Also we recall that $r = (s - a) \tan \frac{1}{2}A$, with other similar formulas.

Thus we have

$$(17) \qquad r_u^2 = \frac{3}{4} u^2 \left(\frac{u - 2}{u + 2}\right)$$

$$(18) \qquad r_b^2 = \frac{3}{4} (u^2 - 4) \,,$$

whence by (11), $r_b = 3r$,

$$(19) \qquad r_c^2 = \frac{3}{4} u^2 \left(\frac{u + 2}{u - 2}\right) \,.$$

The radii of the three escribed circles are easily calculated and the first few values are as follows:

$$(20) \qquad r_a : 0,\ 2,\ \frac{21}{2},\ \frac{130}{3},\ \frac{1164}{7},\ \frac{6878}{11},\ \frac{50795}{13},\ \cdots$$

$$(21) \qquad r_b : 0,\ 3,\ 12,\ 45,\ 168,\ 627,\ 2340,\ \cdots$$

$$(22) \qquad r_c : \infty,\ 6,\ 14,\ \frac{234}{5},\ \frac{679}{4},\ \frac{11946}{19},\ \cdots \,.$$

Relations (16) become

$$(23) \qquad \frac{1}{r_a} + \frac{1}{r_c} = \frac{2}{3r} \,,$$

and

$$(24) \qquad r_a + r_c = 4R - 2r = \frac{6r^2 + 2}{r}$$

the last step following because of the fact that we shall find $R = 2r + 1/2r$.

As a simple example of the check afforded by (23), we have $(n = 5)$

$$\frac{19}{11946} + \frac{11}{6878} = \frac{19}{2\cdot 3\cdot 11\cdot 181} + \frac{11}{2\cdot 19\cdot 181} = \frac{19^2 + 3\cdot 11^2}{2\cdot 3\cdot 11\cdot 19\cdot 181}$$

$$= \frac{361 + 363}{2\cdot 3\cdot 11\cdot 19\cdot 181} = \frac{2}{3\cdot 11\cdot 19} = \frac{2}{3(209)} = \frac{2}{3r} \,.$$

One discerns a Pellian equation in this calculation also.

We may combine (23) and (24) to obtain a product formula, which is

$$r_a r_c = 9r^2 + 3 . \tag{25}$$

The equation

$$x^2 - (r_a + r_c)x + r_a r_c = 0$$

has for roots the radii r_a, r_c, and when we substitute into this equation by means of (24) and (25), we have the equation

$$rx^2 - (6r^2 + 2)x + 9r^3 + 3r = 0 .$$

Solving this by the quadratic formula, we obtain the novel formulas

$$r_a = \frac{3r^2 + 1 - \sqrt{3r^2 + 1}}{r} \tag{26}$$

and

$$r_c = \frac{3r^2 + 1 + \sqrt{3r^2 + 1}}{r} \tag{27}$$

which are rather elegant results, especially since $3r^2 + 1$ is a perfect square.

We turn now to the angles of our triangles. From the functional relations

$$2K = ab \sin C = bc \sin A = ca \sin B , \tag{28}$$

we find (by means of (1))

$$\sin^2 A = \frac{3}{4} \frac{u^2 - 4}{(u + 1)^2} , \tag{29}$$

$$\sin^2 B = \frac{3}{4} \frac{u^2(u^2 - 4)}{(u^2 - 1)^2} , \tag{30}$$

$$\sin^2 C = \frac{3}{4} \frac{u^2 - 4}{(u - 1)^2} . \tag{31}$$

Letting $n \rightarrow \infty$, each of these tends to $3/4$. This agrees with the fact that in an equilateral triangle the three sines would be each $\sqrt{3}/2$. Of course, our special triangle T_n behaves at ∞ as an equilateral triangle insofar as <u>angular</u> measurements are concerned, but never becomes truly an equilateral triangle because the <u>sides</u> never become equal. We may illustrate this behavior in another way. It is well known that the square of the distance between the circumcenter and incenter in any triangle is $R(R - 2r)$. Since, as we have remarked, it can be shown in our case that $R = 2r + 1/2r$ the number in question has the value $R/2r$. It is also known that $R \geq 2r$ in any case. However, $R - 2r = 1/2r$ which can be made as small as we wish by choosing n sufficiently large. (It follows from (12) that r_n is an increasing sequence.) Thus we have

$$\lim_{n\to\infty} (R_n - 2r_n) = 0 \tag{32}$$

and

$$\lim_{n\to\infty} \frac{R_n}{2r_n} = 1 . \tag{33}$$

It follows then that the distance between circumcenter and incenter tends to 1. Only if these two points come together can we speak truly of an equilateral triangle. Of course, in a finite triangle, with R fixed say, then as $2r$ approaches R, $R(R - 2r)$ tends to zero. In our case, however $(R - 2r)^{-1}$ and R increase at the same rate, i.e., $n \to \infty$. The reader will find other peculiarities of T_∞.

Let us agree to write $|P - Q|$ for the distance between points P and Q. Let N = circumcenter; N = orthocenter; I = incenter; G = centroid; M = Nine-point center; A, B, C = vertices. Then we have the following known distance relationships for an arbitrary triangle:

$$|N - H|^2 = 9R^2 - (a^2 + b^2 + c^2) = 9|N - G|^2 = \frac{9}{4}|G - H|^2 ;$$

$$|I = H|^2 = 4R^2 + 2r^2 - \frac{1}{2}(a^2 + b^2 + c^2) ;$$

$$|I - N|^2 = R(R - 2r) ;$$

$$|I - A| \cdot |I - B| \cdot |I - C| = 4r^2R ;$$

$$|G - H| = 2|G - N| ;$$

$$|M - N| = |M - H| = \frac{1}{2}|N - H| .$$

In our special triangles we also have the following:

$$ab + bc + ca = 3u^2 - 1 = 3u_n^2 - 1 = 3u_{2n} + 5 , \tag{34}$$

and

$$a^2 + b^2 + c^2 = 3u^2 + 2 = 3u_{2n} + 8 = 36r^2 + 14 . \tag{35}$$

Thus we have

$$|N - H|^2 = 9\left(2r + \frac{1}{2r}\right)^2 - 36r^2 - 14 = 4 + \frac{9}{4r^2}, \tag{36}$$

and since r increases steadily with n we see that for T_∞ the circumcenter and orthocenter will be two units apart.

Moreover,

$$\begin{aligned} |I - H|^2 &= 4(2r + 1/2r)^2 + 2r^2 - \tfrac{1}{2}(36r^2 + 14) \\ &= 1 - 1/r^2 , \end{aligned}$$

whence in T_∞ the incenter and orthocenter are also one unit apart.

It is then extremely simple to draw the Euler line for T_∞:

N G I = M H

The Euler line from N to H is two units long and the incenter lies on it and in fact coincides with the Nine-Point center. This then gives some idea of the behavior of T_∞.

In Figure 1 is shown the standard location of the common points in an arbitrary finite triangle. The Nine-point circle has quite a history, having been studied as long ago as 1804. It was first called "le cercle des neuf points" by Terquem in 1842 in Vol. 1 of the journal Nouvelles Annales de Mathématiques. The circle has many properties; it passes through the midpoints of the sides and the feet of the altitudes, it is tangent to the inscribed circle; its residue is $\frac{1}{2}R$; it bisects any line segment drawn from the orthocenter to the circumcircle. Thus it has more than nine points associated with it, and has been called an n-point circle, Terquem's circle, the medioscribed circle, the circumscribed midcircle, Feuerbach's circle, etc. A very interesting history has been given by J. S. MacKay [6]. Coxeter [1, p. 18] quotes Daniel Pedoe: "This circle is the first really exciting one to appear in any course on elementary geometry."

We have now to return to a discussion of the circumradius R. From the formula

$$K = \frac{abc}{4R}$$

we have in our case

$$R = R_n = \frac{u^2 - 1}{6r} = \frac{u_n^2 - 1}{6r} = \frac{u_{2n} + 1}{6r}, \tag{37}$$

or also

$$R_n^2 = \frac{(u^2 - 1)^2}{3(u^2 - 4)}. \tag{38}$$

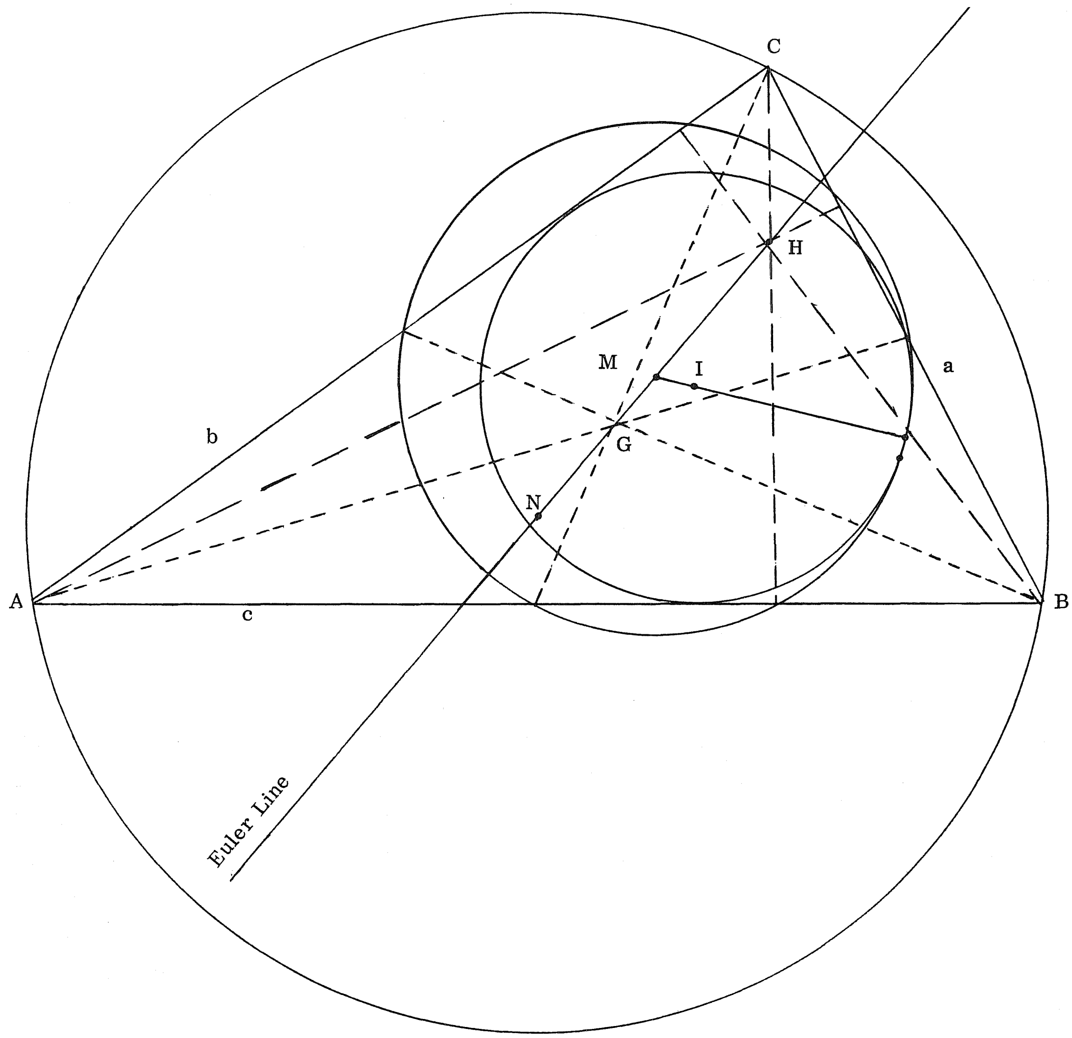

Figure 1

But by (11) we have $u^2 = 12r^2 + 4$, so

$$R^2 = \frac{(12r^2 + 3)^2}{3(12r^2)} = \frac{(4r^2 + 1)^2}{4r^2} ,$$

whence

(39) $$R = 2r + \frac{1}{2r}$$

as we suggested earlier. The first few values of R are

$$\infty,\ \frac{5}{2},\ \frac{65}{8},\ \frac{901}{30},\ \frac{12545}{112},\ \frac{174725}{418},\ \cdots$$

or

$$0+\frac{1}{0},\ 2+\frac{1}{2},\ 8+\frac{1}{8},\ 30+\frac{1}{30},\ 112+\frac{1}{112},\ 418+\frac{1}{418},\ 1560+\frac{1}{1560},\ \cdots$$

It is certainly more interesting, for example, in the triangle 13, 14, 15 to think of the circumradius as $8 + 1/8$ than as $65/8$; this together with the inradius being 4. (We apologize for writing $1/0$ but wish to be suggestive.)

The sequence of numbers 1, 5, 65, 901, 12545, $\cdots$ incidentally, has an interesting recurrence. Now we know that these are just $2r$ times R, so let us define a special sequence by

$$g_n = 2rR = 2r_nR_n . \tag{40}$$

Then $g_n = (u^2 - 1)/3$, but also

$$g_{n+2} = 14g_{n+1} - g_n - 4, \qquad (g_0 = 1,\quad g_1 = 5) . \tag{41}$$

This completes our present discussion of the properties of special number sequences associated with the class of triangles having consecutive integers as sides and having integral areas. The really crucial matter was right at the beginning where it was necessary to set up a criterion for the triangles. It is not enough to guess formula (3) or (5), as we must rule out any other possibility. This we accomplished by setting up the equation (1) and arguing to (2) as a <u>necessary</u> condition. That it is a <u>sufficient</u> condition is clear. Any three consecutive numbers (>1) do generate a real triangle, and sequence (3) turns out to have integral area.

We close by suggesting other possible problems. Let $u \geq 2$ and consider triangles having integral areas and sides $2u - 1$, u, $2u + 1$. Then $s = 5u/2$, and

$$s - a = \frac{1}{2}(u + 2), \qquad s - b = 3u/2, \qquad s - c = \frac{1}{2}(u - 2) .$$

Then

$$K^2 = s(s - a)(s - b)(s - c) = \frac{15u^2(u^2 - 4)}{16} .$$

Again, u must be even. Thus we have evidently to impose the equation

$$u^2 - 15v^2 = 4 . \tag{42}$$

The rest of the discussion is similar to what we presented above.

Again, let the sides be consecutive Fibonacci numbers. Then

$$s = \frac{1}{2}(F_{n-1} + F_n + F_{n+1}) = \frac{1}{2}(F_{n+1} + F_{n+1}) = F_{n+1} ,$$

and

$$s - a = F_n , \quad s - b = F_{n-1} , \quad a - c = 0 .$$

Thus $K = 0$. But this is trivial. No triangle is formed; just a degenerate line segment. It would be of interest to modify the values so as to have some really interesting Fibonacci triangle with integral area. We leave this as a problem for any interested reader. Can one, for instance, make anything interesting with sides $F_m - d$, F_m, $F_m + d$ for suitable values of d? What interesting Pellian equations and recurrences might be associated with a tetrahedron?

REFERENCES

1. H. S. M. Coxeter, Introduction to Geometry, New York, 1961.
2. H. Davenport, The Higher Arithmetic, London, 1952.
3. H. W. Gould, Problem H-37, Fibonacci Quarterly, Vol. 2 (1964), p. 124.
4. R. Hoppe, "Rationales Dreieck," dessen Seiten auf einander folgende ganzen Zahlen sind, Archiv der Mathematik und Physik, Vol. 64 (1880), pp. 441-443. Cf. Jahrbuch über die Fortschritte der Mathematik, Vol. 12 (1880), p. 132.
5. W. J. LeVeque, Topics in Number Theory, Reading, Mass., 1956.
6. J. S. MacKay, "History of the Nine-Point Circle," Proceedings of the Edinburgh Mathematical Society, Vol. 11 (1892/93), pp. 19-57.
7. T. R. Running, Problem 4047, Amer. Math. Monthly, Vol. 49 (1942), p. 479; Solutions by W. B. Clarke and E. P. Starke, ibid., Vol. 51 (1944), pp. 102-104.
8. W. B. Clarke, Problem 65, National Math. Mag., Vol. 9 (1934), p. 63.
9. L. E. Dickson, History of the Theory of Numbers, Washington, D.C., 3 Volumes, 1919-1923. Chelsea Reprint, New York, 1952.

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