A PRIMER FOR THE FIBONACCI NUMBERS
PART XI: MULTISECTION GENERATING FUNCTIONS FOR THE COLUMNS OF PASCAL'S TRIANGLE

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1. INTRODUCTION

Let

$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$

be the generating function for the sequence $\{a_n\}$. Often one desires generating functions which multisect the sequence $\{a_n\}$,

$$G_i(x) = \sum_{j=0}^{\infty} a_{i+mj} x^j, \qquad (i = 0, 1, 2, \cdots, m-1).$$

For the bisection generating functions the task is easy. Let

$$H_1(x^2) = \frac{f(x) + f(-x)}{2},$$

$$H_2(x^2) = \frac{f(x) - f(-x)}{2x};$$

then clearly $H_1(x^2)$ and $H_2(x^2)$ contain only even powers of x so that

$$H_1(x) = \sum_{n=0}^{\infty} a_{2n} x^n \qquad \text{and} \qquad H_2(x) = \sum_{n=0}^{\infty} a_{2n+1} x^n$$

are what we are looking for.

Let us illustrate this for the Fibonacci sequence. Here

$$f(x) = \frac{x}{1 - x - x^2} = \sum_{n=0}^{\infty} F_n x^n;$$

then

$$H_1(x) = \frac{x}{1 - 3x + x^2} = \sum_{n=0}^{\infty} F_{2n} x^n$$

and

$$H_2(x) = \frac{1 - x}{1 - 3x + x^2} = \sum_{n=0}^{\infty} F_{2n+1} x^n .$$

Exercise: Find the bisection generating functions for the Lucas sequence.

Let us find the general multisecting generating functions for the Fibonacci sequence, using the method of H. W. Gould [1]. The Fibonacci sequence enjoys the Binet Form

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \qquad \alpha = \frac{1 + \sqrt{5}}{2}, \qquad \beta = \frac{1 - \sqrt{5}}{2} .$$

Let $f(x) = 1/(1 - x)$; then

$$\begin{aligned}\sum_{n=0}^{\infty} F_{mn+j} x^n &= \frac{\alpha^j f(\alpha^m x) - \beta^j f(\beta^m x)}{\alpha - \beta} \\ &= \frac{1}{\alpha - \beta}\left(\frac{\alpha^j}{1 - \alpha^m x} - \frac{\beta^j}{1 - \beta^m x}\right) \\ &= \frac{\dfrac{\alpha^j - \beta^j}{\alpha - \beta} + (\alpha\beta)^j \dfrac{\alpha^{m-j} - \beta^{m-j}}{\alpha - \beta} x}{1 - (\alpha^m + \beta^m)x + (\alpha\beta)^m x^2} \\ &= \frac{F_j + (-1)^j F_{m-j} x}{1 - L_m x + (-1)^m x^2}, \quad (j = 0, 1, 2, \cdots, m - 1),\end{aligned}$$

since $\alpha\beta = -1$, $\alpha^m + \beta^m = L_m$, and

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} .$$

Exercise: Find the general multisecting generating function for the Lucas sequence.

The same technique can be used on any sequence having a Binet Form. The general problem of multisecting a general sequence rapidly becomes very complicated according to Riordan [2], even in the classical case.

2. COLUMN GENERATORS OF PASCAL'S TRIANGLE

The column generators of Pascal's left-justified triangle [3], [4], [5], are

$$G_k(x) = \frac{x^k}{(1-x)^{k+1}} = \sum_{n=0}^{\infty} \binom{n}{k} x^n, \qquad k = 0, 1, 2, \cdots.$$

We now seek generating functions which will m-sect these,

$$G_i(m, k; x) = \sum_{n=0}^{\infty} \binom{i + k + mn}{k} x^{n+k+1}, \qquad (i = 0, 1, \cdots, m-1).$$

We first cite an obvious little lemma.

Lemma 1.

$$\binom{n}{k} = \sum_{j=1}^{m} \binom{n-j}{k-1} + \binom{n-m}{k}.$$

Definition. Let $G_{i,k}(x)$, $i = 0, 1, 2, \cdots, m-1$, be the m generating functions

$$G_{i,k}(x) = \sum_{n=0}^{\infty} \binom{i + k + mn}{k} x^{i+mn+k}.$$

Lemma 2.

$$G_{i,k+1}(x) = \frac{xG_{i,k}(x) + x^2G_{i-1,k}(x) + \cdots + x^mG_{i-m+1,k}(x)}{1 - x^m}.$$

The proof follows easily from Lemma 1.

Let

$$(1 + x + x^2 + \cdots + x^{m-1})^n = \sum_{j=0}^{n(m-1)} \binom{n}{j}_m x^j$$

define the row elements of the m-nomial triangle. Further, let

$$f_i(m, k; x) = \sum_{j=0} \binom{k}{i + jm}_m x^j, \qquad i = 0, 1, \cdots, m-1,$$

where j is such that $i + jm \le k(m-1)$. These are multisecting polynomials for the rows of the m-nomial triangle. Now, we can state an interesting theorem:

Theorem. For $i = 0, 1, 2, \cdots, m-1$,

$$G_i(m,k;\,x) = \frac{x^{k+i} f_i(m,k;\,x)}{(1-x)^{k+1}} .$$

Proof. Recall first that the m-nomial coefficients obey

$$\binom{n}{r}_m = \binom{n-1}{r}_m + \binom{n-1}{r-1}_m + \cdots + \binom{n-1}{r-m+1}_m$$

where the lower arguments are non-negative and less than or equal to $n(m-1)$.

Clearly, for $k = 0$, from the definition just before Lemma 2,

$$G_{i,0}(x) = \frac{x^i}{1-x^m}, \qquad i = 0, 1, 2, \cdots, m-1 .$$

Assume now that

$$G_{i,k}(x) = \frac{x^{k+i} f_i(m,k;\,x^m)}{(1-x^m)^{k+1}}$$

for $i = 0, 1, 2, 3, \cdots, (m-1)$. From Lemma 2,

$$G_{i,k+1}(x) = \frac{xG_{i-1,k}(x) + \cdots + x^m G_{i-m+1,k}(x)}{1-x^m} .$$

Thus,

$$G_{i,k+1}(x) = \frac{\sum_{s=0}^{m-1}\left(\sum_{j=0}\binom{k}{i-s+jm}_m\right)x^{k+(i-s)+s+jm+1}}{(1-x^m)^{k+2}}$$

$$= \frac{\sum_{j=0}\left(\sum_{s=0}^{m-1}\binom{k}{i-s+jm}_m\right)x^{k+1+i+jm}}{(1-x^m)^{k+2}}$$

$$= \frac{x^{k+1+i}\sum_{j=0}\binom{k+1}{i+jm}_m x^{jm}}{(1-x^m)^{k+2}}$$

$$= \frac{x^{k+1+i} f_i(m,k;\,x^m)}{(1-x^m)^{k+2}} .$$

This completes the induction.

The x^{k+1+i} merely position the column generators. Here the non-zero entries are separated by $m - 1$ zeros. To get rid of the zeros, let

$$G_i(m,k;x) = \frac{x^{k+i} f_i(m,k;x)}{(1-x)^{k+1}}$$

for $i = 0, 1, 2, \cdots, m-1$. This concludes the proof of the theorem.

If we write this in the form

$$G_i(m,k;x) = \sum_{j=0}^{\infty} \binom{i + jm + k}{k} x^{j+k+1} = \frac{\sum_{j=0} \binom{k}{i+jm}_m x^{k+i+j}}{(1-x)^{k+1}}$$

it emphasizes the relation of the multisection of the k^{th} column of Pascal's triangle and the multisection of the k^{th} row of the m-nomial triangle.

3. A NEAT GENERATING FUNCTION

Lemma 3

$$\binom{n}{k} = \sum_{j=0}^{r} \binom{r}{j}\binom{n-r}{k-j}$$

This is easy to prove by starting with

$$\begin{aligned} \binom{n}{k} &= \binom{n-1}{k} + \binom{n-1}{k-1} \\ \text{(A)} \qquad &= \binom{n-2}{k} + \binom{n-2}{k-1} + \binom{n-2}{k-1} + \binom{n-2}{k-2} \\ &= 1 \cdot \binom{n-2}{k} + 2 \cdot \binom{n-2}{k-1} + 1 \cdot \binom{n-2}{k-2} . \end{aligned}$$

Apply (A) to each term on the right repeatedly.

Now let $H_i(m,k;x)$ m-sect the k^{th} column of Pascal's triangle $(i = 0, 1, 2, \cdots, m-1)$; then, using Lemma 3, it follows that

Lemma 4

$$H_i(m,k;x) = \frac{x}{1-x} \sum_{j=1}^{m} \binom{m}{j} H_i(m,k-j;x) .$$

The results using the method of Polya for small m and i seem to indicate the following [3].

Theorem. The generating functions for the rising diagonal sums of the rows of Pascal's triangle $i + jm$ (all other rows are deleted) are given by

$$H_i(x) = \frac{(1+x)^i}{1 - x(1+x)^m}, \qquad i = 0, 1, \cdots, m-1 .$$

Exercise: Show that

$$\sum_{i=0}^{m-1} x^i H_i(x^m) = \frac{1}{1 - x(1+x^m)} .$$

This is a necessary condition which now makes the theorem plausible. These are the generalized Fibonacci numbers obtained as rising diagonal sums from Pascal's triangle, beginning in the left-most column and going over 1 and up m 3 . The theorem is proved by careful examination of its meaning with regards to Pascal's triangle as follows:

$$\frac{(1+x)^i}{1 - x(1+x)^m} = \sum_{n=0}^{\infty} x^n (1+x)^{mn+i} = \sum_{n=0}^{\infty} \sum_{j=0}^{n} \binom{m(n-j)+i}{j} x^n ,$$

Recall that $\binom{n}{k} = 0$ if $0 \le n \le k$.

ILLUSTRATION

$$\begin{array}{lll}
n = 0 & x^0(1+x)^{0+1} = & 1 + x \\
n = 1 & x^1(1+x)^{2+1} = & x + 3x^2 + 3x^3 + x^4 \\
n = 2 & x^2(1+x)^{4+1} = & x^2 + 5x^3 + 10x^4 + 10x^5 + 5x^6 + x^7 \\
n = 3 & x^3(1+x)^{6+1} = & x^3 + 7x^4 + 21x^5 + \cdots \\
\cdots & \cdots & \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
\text{Sum:} & & 1 + 2x + 4x^2 + 9x^3 + 19x^4 + \cdots
\end{array}$$

Here, $m = 2$ and $i = 1$. Now, write a left-justified Pascal's triangle. Form the sequence of sums of elements found by beginning in the left-most column and proceeding right one and up two throughout the array: 1, 1, 1, 2, 3, 4, 6, 9, 13, 19, $\cdots$. Notice that the coefficients of successive powers of x give every other term in that sequence.

The general problem of finding generating functions which multisect the column generators of Pascal's triangle has been solved by Nilson [6], although interpretation of the numerator polynomial coefficients has not been achieved as in our last few theorems.

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