ADVANCED PROBLEMS AND SOLUTIONS

Edited by
RAYMOND E. WHITNEY
Lock Haven State College, Lock Haven, Pennsylvania

Send all communications concerning Advanced Problems and Solutions to Raymond E. Whitney, Mathematics Department, Lock Haven State College, Lock Haven, Pennsylvania 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, solutions should be submitted on separate signed sheets within two months after publication of the problems.

H-215 Proposed by Ralph Fecke, North Texas State University, Denton, Texas

a. Prove

$$\sum_{i=n}^{n+2} 2^i P_i \equiv 0 \pmod 5$$

for all positive integers, n; P_i is the i^{th} term of the Pell sequence, $P_1 = 1$, $P_2 = 2$, $P_{n+1} = 2P_n + P_{n-1}$ $(n \geq 2)$.

b. Prove $2^n L_n \equiv 2 \pmod{10}$ for all positive integers n; L_n is the n^{th} term of the Lucas sequence.

H-216 Proposed by Guy A. R. Guillotte, 229 St. Joseph Blvd., Cowansville, Quebec, Canada.

Let G_m be a set of rational integers and

$$\sum_{n=1}^{\infty}\left[\log_e\left(\sum_{m=0}^{\infty}\frac{G_m}{(m)!\,(F_{2n+1})^m}\right)\right] = \frac{\pi}{4}.$$

Find a formula for G_m.

H-217 Proposed by S. Krishna, Orissa, India.

A. Show that

$$2^{4n-4x-4}\binom{2x+2}{x+1} \equiv \binom{4n-2x-2}{2n-x-1} \pmod{4n+1},$$

where n is a positive integer and $-1 \leq x \leq 2n - 1$ and x is an integer, also.

B. Show that

$$2^{4n-4x-6}\binom{2x+4}{x+2} + \binom{4n-2x-2}{2n-x-1} \equiv 0 \pmod{4n+3},$$

where n is a positive integer and $-2 \leq x \leq 2n - 1$ and x is an integer, also.

H-218 Proposed by V. E. Hoggatt, Jr., San Jose State University, San Jose, California.

Let

$$A = \begin{pmatrix} 1 & 0 & 0 & & \cdots \\ 0 & 1 & 0 & & \cdots \\ 0 & 1 & 1 & 0 & \cdots \\ \cdots & 2 & 1 & & \cdots \end{pmatrix}_{n\times n}$$

represent the matrix which corresponds to the staggered Pascal Triangle and

$$B = \begin{pmatrix} 1 & 1 & 1 & 1 & \cdots \\ 1 & 2 & 3 & 4 & \cdots \\ 1 & 3 & 6 & 10 & \cdots \end{pmatrix}_{n\times n}$$

represent the matrix which corresponds to the Pascal Binomial Array. Finally, let

$$C = \begin{pmatrix} 1 & 1 & 1 & 1 & \cdots \\ 1 & 2 & 3 & 4 & \cdots \\ 2 & 5 & 9 & 14 & \cdots \end{pmatrix}_{n\times n}$$

represent the matrix corresponding to the Fibonacci Convolution Array. Prove $AB = C$.

H-219 Proposed by Paul Bruckman, University of Illinois, Urbana, Illinois.

Prove the identity

$$(-1)^n \binom{x}{n} \sum_{i=0}^{n} \binom{n}{i} (-2)^i \cdot \frac{x-n}{x-i} = \sum_{i=0}^{n} \binom{x}{i},$$

where

$$\binom{x}{i} = \frac{x(x-1)(x-2)\cdots(x-i+1)}{i!}$$

(x not necessarily an integer).

H-220 Proposed by L. Carlitz, Duke University, Durham, North Carolina.

Show that

$$\sum_{k=0}^{\infty} \frac{a^k z^k}{(z)_{k+1}} = \sum_{r=0}^{\infty} \frac{a^r q^{r^2} z^{2r}}{(z)_{r+1}(az)_{r+1}} ,$$

where

$$(z)_n = (1 - z)(1 - qz) \cdots (1 - q^{n-1}z), \qquad (z)_0 = 1 .$$

SOLUTIONS

ANOTHER PIECE

H-125 Proposed by Stanley Rabinowitz, Far Rockaway, New York.

Define a sequence of positive integers to be left-normal if given any string of digits, there exists a member of the given sequence beginning with this string of digits, and define the sequence to be right-normal if there exists a member of the sequence ending with the string of digits.

Show that the sequence whose n^{th} terms are given by the following are left-normal but not right-normal.

a. $P(n)$, where $P(x)$ is a polynomial function with integral coefficients.

b. P_n, where P_n is the n^{th} prime.

c. $n!$

d. F_n, where F_n is the n^{th} Fibonacci number.

Partial solution by R. Whitney, Lock Haven State College, Lock Haven, Pennsylvania.

Using a theorem of R. S. Bird*, one may show that each of the above is left-normal. If

$$\lim_{n \to \infty} \frac{S_{n+1}}{S_n} = \theta ,$$

where $\theta = 1$ or θ is not a rational power of 10 or if

$$\lim_{n \to \infty} \frac{S_{n+1}}{S_n} = \infty \quad \text{and} \quad \lim_{n \to \infty} \frac{S_n S_{n+2}}{S_{n+1}^2} = 1 ,$$

then $\{S_n\}_{n=1}^{\infty}$ is left-normal (extendable in base 10).

a. $$\lim_{n \to \infty} \frac{P(n + 1)}{P(n)} = 1 ,$$

hence $\{P(n)\}_{n=1}^{\infty}$ is left normal.

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*R. S. Bird, "Integers with Given Initial Digits," Amer. Math. Monthly, Apr. 1972, pp. 367-370.

c. $$\lim_{n\to\infty} \frac{(n+1)!}{n!} = \infty \qquad \text{and} \qquad \lim_{n\to\infty} \frac{n!(n+2)!}{((n+1)!)^2} = 1,$$

thus $\{n!\}_{n=1}^{\infty}$ is left-normal.

d. $$\lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \frac{1+\sqrt{5}}{2},$$

thus $\{F_n\}_{n=1}^{\infty}$ is left-normal, also. The only question which remains is the demonstration that the sequences are not right-normal.

(c) is easy, since $n!$ is divisible by 4 for $n \geq 4$. Clearly no factorial, then, ends in 21, in particular. The final problem which remains is the question of right-normality for (a) and (d).

COMMENT ON H-174

H-174 Proposed by Daniel W. Burns, Chicago, Illinois.

Let k be any non-zero integer and $\{S_n\}_{n=1}^{\infty}$ be the sequence defined by $S_n = nk$.

Define the Burns Function, $B(k)$, as follows: $B(k)$ is the minimal value of n for which each of the ten digits, $0, 1, \cdots, 9$ have occurred in at least one S_m where $1 \leq m \leq n$. For example, $B(1) = 10$, $B(2) = 45$. Does $B(k)$ exist for all k? If so, find an effective formula or algorithm for calculating it.

Comment by R. E. Whitney, Lock Haven State College, Lock Haven, Pennsylvania.

Using the theorem by Bird, referred to in the above H-125, we have

$$\lim_{n\to\infty} \frac{(n+1)k}{nk} = 1 \qquad \text{and} \qquad \{nk\}_{n=1}^{\infty}$$

is left-normal, or extendable in base 10. Thus, in particular, the sequence $123 \cdots 90$ occurs at $\{nk\}_{n=1}^{\infty}$. The existence of $B(k)$ now follows by well ordering. One can show that $B(2k) > B(k)$ and other assorted inequalities.

ANOTHER REMARK

H-182 Proposed by S. Krishna, Orissa, India.

Prove or disprove:

(i) $$\sum_{k=1}^{m} \frac{1}{k^2} \equiv 0 \pmod{2m+1},$$

and

(ii) $$\sum_{k=1}^{m} \frac{1}{(2k-1)^2} \equiv 0 \pmod{2m+1}$$

when $2m + 1$ is prime and larger than 3.

Comment by R. E. Whitney, Lock Haven State College, Lock Haven, Pennsylvania.

It is well known* that

$$\text{(iii)} \qquad \sum_{k=1}^{2m} \frac{1}{k^2} \equiv 0 \pmod{2m + 1},$$

when $2m + 1$ is prime and larger than 3.

Set

$$\sigma_1 = \sum_{k=1}^{m} \frac{1}{k^2} \quad \text{and} \quad \sigma_2 = \sum_{k=1}^{m} \frac{1}{(2k - 1)^2} .$$

Using (iii), we have

$$1/4\sigma_1 + \sigma_2 \equiv 0 \pmod{2m + 1}$$

or

$$\sigma_1 + 4\sigma_2 \equiv 0 \pmod{2m + 1} .$$

From the above, it follows that (i) and (ii) are equivalent.

NOT THIS TIME

H-193 Proposed by Edgar Karst, University of Arizona, Tucson, Arizona.

Prove or disprove: If $x + y + z = 2^{2n+1} - 1$ and $x^3 + y^3 + z^3 = 2^{6n+1} - 1$, then $6n + 1$ and $2^{6n+1} - 1$ are primes.

Solution by Paul Bruckman, University of Illinois, Urbana, Illinois.

The following particular solution is sufficient to prove that the conjecture is false. If $(x,y,z) = (1,\ 2^{2n} - 2^n - 1,\ 2^{2n} + 2^n - 1)$, it is easily verified that this solution satisfies the requirements (a) $x + y + z = 2^{2n+1} - 1$, and (b) $x^3 + y^3 + z^3 = 2^{6n+1} - 1$. Moreover, this is true for all non-negative integers n, in particular when $n = 4$, i.e., $6n + 1 = 25$, which is not prime. It might be of interest to determine if any other solutions, not necessarily Diophantine, exist, although this was not attempted here.

Also solved by T. Carroll, D. Finkel, and D. Zeitlin.

The editor would like to acknowledge solutions to the following Problems:

H-173, H-176 Clyde Bridger; H-187 K. Wayland and D. Priest, E. Just, G. Wulczyn, and J. Ire; H-190 L. Frohman, R. Fecke, L. Carlitz, P. Smith; H-191 L. Carlitz; H-192 L. Carlitz, D. Zeitlin, and P. Bruckman.

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*Hardy and Wright, The Theory of Numbers, Oxford University Press, London, 1962, p. 90.