THE FIRST SOLUTION OF THE CLASSICAL EULERIAN MAGIC CUBE PROBLEM OF ORDER TEN

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In this paper for the first time three Latin cubes of the tenth order have been superimposed to form an Eulerian cube. A Latin cube of the tenth order is defined as a cube of 1000 cells (in ten rows, ten columns, and ten files) in which 1000 numbers consisting of 100 zeros, 100 ones, $\cdots$, 100 nines, are arranged in the cells so that the ten numbers in each row, each column, and each file are different.

In this paper, we actually solved two problems, since in addition to having solved the Eulerian cube of order ten, we have also made the cube magic (for the first time). A magic cube is such that the ten cells in each diagonal (or "diameter") and in every row, every file, and every column is the same — namely, 4995 (see [1]).

In what follows, it will be noted that each of the ten SQUARES contain 100 cells and each cell contains a three-digit number. Now, if we delete the third digit on the right side in each and every cell, it is easily verified that each of the ten SQUARES has become pairwise orthogonal.

In 1779, Euler conjectured that no pair of orthogonal squares exist for $n \equiv 2 \pmod 4$. Then in 1959, the Euler conjecture was shown to be incorrect by the remarkable mathematics of Bose, Shrikande and Parker [2]. Recently (in 1972) Hoggatt and this author extended Bose, Shrikande and Parker's work by finding a way to make the 10×10 square pairwise orthogonal as well as magic. For a square to be magic, each of the two diagonals must have the same sum as in every row and in every column — namely (since we are considering the sum of ten cells with two digits in each cell), 495 (see [3]).

Let us label the cells in each square as follows: (row, column, square number) = (r, c, s) = some number in a cell. For example, the number 763 in Square Number 0 reads $763 = (0,0,0)$, or say we wish to consider the number 338 in Square Number 1: we then write $338 = (6, 2, 1)$.

THEN THE SUM OF EACH DIAGONAL (OR "DIAMETER") IN THE FOLLOWING FOUR-DIAMETER MAGIC CUBE IS, RESPECTIVELY,

$$\sum_{r,c,s=0}^{9} (r,c,s) = \sum_{r,c,s=0}^{9} (9-r,c,s) = \sum_{r,c,s=0}^{9} (r,9-c,s) = \sum_{r,c,s=0}^{9} (9-r,9-c,s) = 4995.$$

Now, let us define a magic route as that path which goes through ten different squares and passes through one cell in each square and no two cells that the route traverses are in the same file, and the sum total of the numbers in the ten cells that make up this magic route equals 4995.

Then it may be easily shown that any cell in the cube begins a magic route. For example:

$$(4,2,0) + (8,4,1) + (6,0,2) + (0,7,3) + (5,8,4) + (9,5,5) + (1,3,6) + (3,1,7) + (2,6,8) + (7,9,9) = 4995.$$

For the convenience of the reader, we list, respectively, the numbers represented by notation above — 754, 321, 737, 575, 762, 003, 480, 396, 648, and 319.)

Note: The general method of how to find magic routes in singly-even magic cubes (except 2 and 6) will be given in the forthcoming paper mentioned above.

SQUARE NUMBER 0

763	886	540	979	015	428	601	354	232	197
279	963	097	654	832	301	728	186	440	515
897	340	463	201	579	632	154	915	028	786
140	454	901	063	628	715	879	297	586	332
932	228	754	815	163	086	597	401	379	640
328	697	132	740	486	563	215	079	954	801
554	032	286	128	701	997	363	840	615	479
415	779	828	532	397	240	986	663	101	054
686	501	315	497	254	179	040	732	863	928
001	115	679	386	940	854	432	528	797	263

SQUARE NUMBER 1

472	138	264	085	793	616	947	821	359	500
385	072	700	921	159	847	416	538	664	293
100	864	672	347	285	959	521	093	716	438
564	621	047	772	916	493	185	300	238	859
059	316	421	193	572	738	200	647	885	964
816	900	559	464	638	272	393	785	021	147
221	759	338	516	447	000	872	164	993	685
693	485	116	259	800	364	038	972	547	721
938	247	893	600	321	585	764	459	172	016
747	593	985	838	064	121	659	216	400	372

SQUARE NUMBER 2

190	924	771	313	808	565	289	637	446	052
413	390	852	237	946	689	165	024	571	708
952	671	590	489	713	246	037	308	865	124
071	537	389	890	265	108	913	452	724	646
346	465	137	908	090	824	752	589	613	271
665	252	046	171	524	790	408	813	337	989
737	846	424	065	189	352	690	971	208	513
508	113	965	746	652	471	324	290	089	837
224	789	608	552	437	013	871	146	990	365
889	008	213	624	371	937	546	765	152	490

SQUARE NUMBER 3

987	250	823	431	649	002	794	575	118	366
131	487	666	775	218	594	902	350	023	849
266	523	087	194	831	718	375	449	602	950
323	075	494	687	702	949	231	166	850	518
418	102	975	249	387	650	866	094	531	723
502	766	318	923	050	887	149	631	475	294
875	618	150	302	994	466	587	223	749	031
049	931	202	818	566	123	450	787	394	675
750	894	549	066	175	331	623	918	287	402
694	349	731	550	423	275	018	802	966	187

SQUARE NUMBER 4

606	541	355	727	434	999	010	162	883	278
827	706	478	062	583	110	699	241	955	334
578	155	906	810	327	083	262	734	499	641
255	962	710	406	099	634	527	878	341	183
783	899	662	534	206	441	378	910	127	055
199	078	283	655	941	306	834	427	762	510
362	483	841	299	610	778	106	555	034	927
934	627	599	383	178	855	741	006	210	462
041	310	134	978	862	227	455	683	506	799
410	234	027	141	755	562	983	399	678	806

SQUARE NUMBER 5

525	069	488	842	157	230	376	903	691	714
642	825	114	303	091	976	530	769	288	457
014	988	225	676	442	391	703	857	130	569
788	203	876	125	330	557	042	614	469	991
891	630	503	057	725	169	414	276	942	388
930	314	791	588	269	425	657	142	803	076
403	191	669	730	576	814	925	088	357	242
257	542	030	491	914	688	869	325	776	103
369	476	957	214	603	742	188	591	025	830
176	757	342	969	888	003	291	430	514	625

SQUARE NUMBER 6

044	312	136	698	961	777	453	280	505	829
598	644	929	480	305	253	077	812	736	161
329	236	744	553	198	405	880	661	977	012
836	780	653	944	477	061	398	529	112	205
605	577	080	361	844	912	129	753	298	436
277	429	805	036	712	144	561	998	680	353
180	905	512	877	053	629	244	336	461	798
761	098	377	105	229	536	612	444	853	980
412	153	261	729	580	898	936	005	344	677
953	861	498	212	636	380	705	177	029	544

SQUARE NUMBER 7

239	773	617	104	582	351	868	096	920	445
904	139	545	896	720	068	251	473	317	682
745	017	339	968	604	820	496	182	551	273
417	396	168	539	851	282	704	945	673	020
120	951	296	782	439	573	645	368	004	817
051	845	420	217	373	639	982	504	196	768
696	520	973	451	268	145	039	717	882	304
382	204	751	620	045	917	173	839	468	596
873	668	082	345	996	404	517	220	739	151
568	482	804	073	117	796	320	651	245	939

SQUARE NUMBER 8

311	495	909	556	270	884	122	748	067	633
056	511	233	148	467	722	384	695	809	970
433	709	811	022	956	167	648	570	284	395
609	848	522	211	184	370	456	033	995	767
567	084	348	470	611	295	933	822	756	109
784	133	667	309	895	911	070	256	548	422
948	267	095	684	322	533	711	409	170	856
870	356	484	967	733	009	595	111	622	248
195	922	770	833	048	656	209	367	411	584
222	670	156	795	509	448	867	984	333	011

SQUARE NUMBER 9

858	607	092	260	326	143	535	419	774	981
760	258	381	519	674	435	843	907	192	026
681	492	158	735	060	574	919	226	343	807
992	119	235	358	543	826	660	781	007	474
274	743	819	626	958	307	081	135	460	592
443	581	974	892	107	058	726	360	219	635
019	374	707	943	835	281	458	692	526	160
126	860	643	074	481	792	207	558	935	319
507	035	426	181	719	960	392	874	658	243
335	926	560	407	292	619	174	043	881	758

REFERENCES

1. Ball and Coxeter, Mathematical Recreations and Essays, Macmillan, 1962. Refer to Chapters 6 and 7, for some history on these classical problems, especially page 217, where Professor Coxeter points out that there are no known rules for constructing magic cubes of singly-even order.
2. This author learned of the remarkable results of Bose, Shrikande and Parker from two sources:
   a. Ball and Coxeter, Mathematical Recreations and Essays, Macmillan, 1962, p. 191.
   b. On August 29, 1972, in the lobby of the mathematics building at Dartmouth College, Hanover, New Hampshire, I had the privilege of viewing the magnificent 10 × 10 pairwise orthogonal mosaic square greeting you as you enter. The mosaic square was placed in this prominent position in honor of their genius.
3. Arkin and Hoggatt, "The Arkin-Hoggatt Game," presented in person at the Seventy-Seventh Summer Meeting, Dartmouth College, Hanover, New Hampshire, Aug. 29 - Sept. 1, 1972; appeared in the Notices of the Amer. Math. Soc., Vol. 19, No. 5, Issue 139, Aug. 1972, p. A-619, under the number 695-05-8.

NOTE: This paper was presented in person at Brown University, Providence, Rhode Island, 10/28/72; appeared in the AMS Notices, Vol. 19, No. 6, Issue 140, 10/72, p. A-728, under the number 697-A2.

REMARK. No computing machine of any type was used to get the results of this paper.