ON SOLVING NON - HOMOGENEOUS LINEAR DIFFERENCE EQUATIONS

MURRAY S. KLAMKIN
Scientific Research Staff, Ford Motor Company, Dearborn, Michigan

In a recent paper, Weinshenk and Hoggatt [1] gave two methods for obtaining the general solution of the difference equation

$$C_{n+2} = C_{n+1} + C_n + n^m . \tag{1}$$

One method is by expansion and the other by operators. However, in the latter method there are still some open convergence questions. Here we give another method which is equivalent to one of the operator methods but which avoids the convergence question. It will be valid for any linear difference equation with constant coefficients and with any non-homogeneous term on the right-hand side. The solution will be given in terms of the solutions of the corresponding homogeneous equation.

We consider the equation

$$L(E)A_n = G_n , \tag{2}$$

where the linear operator is given by

$$L(E) = E^r + a_1E^{r-1} + a_2E^{r^2} + \cdots + a_r$$

and the a_i's are constants. The corresponding homogeneous equation, $L(E)A_n = 0$, can be solved in the standard way in terms of the roots of $L(x) = 0$. We will denote a solution of the homogeneous equation by the sequence $\{B_i\}$ and for simplicity we will assume that the initial conditions on the B_i's are such that

$$\frac{1}{1 + a_1x + a_2x^2 + \cdots + a_rx^r} = B_0 + B_1x + B_2x^2 + \cdots . \tag{3}$$

If we had chosen arbitrary initial conditions for the B_i's, then the numerator (1) on the left-hand side would have been replaced by some polynomial entailing a further calculation subsequently. This procedure is analogous to solving linear non-homogeneous differential equations. One first solves the homogeneous equation subject to quiescent conditions and then obtains the general solution by a convolution in terms of the non-homogeneous term and the latter solution.

To solve (2), we first write down a generating function of the solution, i.e.,

$$A(x) = A_0 + A_1x + A_2x^2 + \cdots + A_rx^r + \cdots .$$

Then

$$a_1xA(x) = a_1A_0x + a_1A_1x^2 + \cdots + a_1A_{r-1}x^r + \cdots ,$$
$$a_2x^2A(x) = a_2A_0x^2 + \cdots + a_2A_{r-2}x^r + \cdots ,$$
$$\vdots \qquad\qquad \vdots$$
$$a_rx^rA(x) = a_rA_0x^r + \cdots .$$

Adding:

$$A(x)(1 + a_1x + a_2x^2 + \cdots + a_rx^r) = S_0 + S_1x + S_2x^2 + \cdots + S_{r-1}x^{r-1} + G_0x^r + G_1x^{r+2} + G_2x^{r+2} + \cdots ,$$

where

$$S_i = a_iA_0 + a_{i-1}A_1 + A_{i-2}A_2 + \cdots + a_0A_i \qquad (a_0 = 1) .$$

Now using (3) and carrying out the multiplication, we obtain the convolution

$$A_n = \{S_0B_n + S_1B_{n-1} + S_2B_{n-2} + \cdots + S_{r-1}B_{n-r-1}\} + \{G_0B_{n-r} + G_1B_{n-r-1} + G_2B_{n-r-2} + \cdots + G_{n-r}B_0\} \qquad (n \geq r) . \tag{4}$$

The top part of the right-hand side of (4) corresponds to the complementary (homogeneous) solution of (2) whereas the bottom part corresponds to the particular solution. It is to be noted that the method is valid even if the non-homogeneous right-hand side of (2) is part of the complementary solution (i.e., if $L(E)G_n = 0$).

We now apply this technique to (1). One complementary solution of (1) is of course the Fibonacci sequence 1, 1, 2, 3, $\cdots$. Thus,

$$\frac{1}{1 - x - x^2} = F_1 + F_2x + F_3x^3 + \cdots .$$

Solution (4) now becomes

$$C_n = C_0F_{n+1} + (C_0 + C_1)F_n + F_1(n-2)^m + F_2(n-3)^m + \cdots + F_{n-2}(1)^m$$

or

$$C_n = C_0F_{n-1} + C_1F_n + \sum_{i=1}^{n-1} F_i(n - i - 1)^m \qquad (n \geq 2) .$$

This corresponds to the solution in [1] provided a stopping rule is used there.

In their concluding remarks, the authors of [1] raise the question of determining conditions under which their operational methods for obtaining a particular solution are valid. They point out the example of D. Lind that if $C_{n+1} - C_n + n$ were to be solved by their operational method, one would obtain

$$C_n = \frac{-1}{1 - E}\{n\} = -\sum_{k=0}^{\infty} E^k n , \tag{5}$$

which diverges unless some stopping rule is involved. However, the divergent solution can be justified if one considers its analytic continuation. First replace n by n^s where $R_e(s) > 1$.

Then in terms of the Riemann Zeta function,

$$-C_n = \frac{1}{n^s} + \frac{1}{n^{s+1}} + \frac{1}{n^{s+2}} + \cdots$$

or

$$C_n = \left\{\frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{(n-1)^s}\right\} - \zeta(s) \; .$$

However, the zeta function can be analytically continued for $R_e(s) < 1$ and for negative integers it is given by [2]

$$\zeta(-2m) = 0, \qquad \zeta(1-2m) = (-1)^m B_m/(2m), \qquad m = 1, 2, 3, \cdots,$$

$$\zeta(0) = -1/2 \qquad (B_m \text{ are the Bernoulli numbers}).$$

Now letting $s = -1$ above, gives the valid particular solution

$$C_n = (1 + 2 + 3 + \cdots + n - 1) - \zeta(-1) \; .$$

Since the constant $\zeta(-1)$ satisfies the homogeneous equation, it can be deleted.

REFERENCES

1. R. J. Weinshenk and V. E. Hoggatt, Jr., "On Solving $C_{n+2} = C_{n+1} + C_n + n^m$ by Expansions and Operators," Fibonacci Quarterly, Vol. 8, No. 1, 1970, pp. 39-48.
2. C. N. Watson, A Course in Modern Analysis, Cambridge University Press, Cambridge, 1946, pp. 267-268.

[Continued from page 162.]

ERRATA

Please make the following correction to "A New Greatest Common Divisor Property of the Binomial Coefficients," appearing on p. 579, Vol. 10, No. 6, Dec. 1972:

On page 584, last equation, for

$$\binom{n+n}{k+a} \qquad \text{read} \qquad \binom{n+a}{k+a}.$$

In "Some Combinatorial Identities of Bruckman," appearing on page 613 of the same issue, please make the following correction.

On the right-hand side of Eq. (12), p. 615, for

$$\frac{2k}{2k+1} \quad \text{read} \quad \frac{2^k}{2k+1} \; .$$