INTERSECTIONS OF LINES CONNECTING TWO PARALLEL LINES

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The purpose of this note is to show that the geometrical method used by the author [1] in proving that the sum of the first n positive integers is $\frac{1}{2}n(n+1)$ also can be used in proving the following result. Given two parallel lines with p points on the first, and q points on the second. Suppose each of the p points is joined by a straight line to each of the q points. Assume that between the parallel lines, no more than two lines intersect at any point. Then the lines joining the points have $\frac{1}{4}pq(p-1)(q-1)$ intersections between the parallel lines.

A proof of this result is as follows:

Label the p points $a_1, a_2, \cdots, a_p$ so that if the index j is greater than the index i, the directed line segment from a_i to a_j is in the same direction for each choice of i and j, $i, j = 1, 2, \cdots, p$ and $i < j$. (Thus the labeling is, for example, from left to right or bottom to top.) See Fig. 1. Label the q points $b_1, b_2, \cdots, b_q$ in a similar manner and so that for $i < j$, the directed line segment from b_i to b_j is in the opposite direction as that from a_1 to a_p. Denote by (a_i, b_j), $i = 1, 2, \cdots, p$ and $j = 1, 2, \cdots, q$ the line between a_i and b_j.

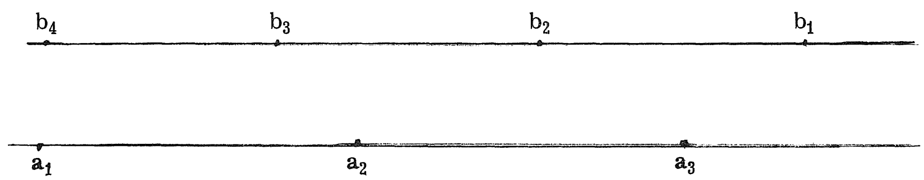

Fig. 1 $(p = 3, \ q = 4)$

Generally we shall place the pq lines sequentially in a certain order, to be specified, and count the number of intersections which arise. The order of placement is lexicographic:

$$(a_1,b_1),\ (a_1,b_2),\cdots,(a_1,b_q),(a_2,b_1),\cdots,(a_2,b_q),\cdots,(a_k,b_1),\cdots,(a_k,b_q),\cdots,(a_p,b_1),\cdots,(a_p,b_q).$$

The first set of lines $(a_1,b_1), \cdots, (a_1,b_q)$ contributes no intersections. See Fig. 2.

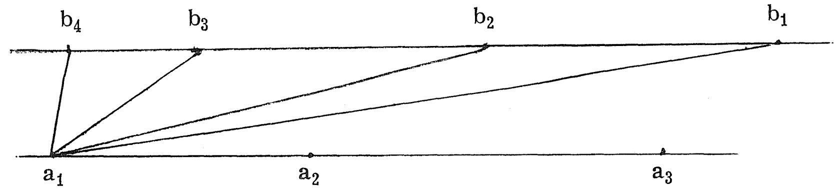

Fig. 2 $(p = 3, \ q = 4)$

Considering the second set of q lines $(a_2, b_1), \cdots, (a_2, b_k), \cdots, (a_2, b_q)$, none of them intersect with each other and (a_2, b_k) intersects with $(k - 1)$ previously placed lines:

$$(a_1, b_1), (a_1, b_2), \cdots, (a_1, b_{k-1}).$$

Thus these q lines contribute

$$(1 + 2 + \cdots + q - 1) = \frac{q(q - 1)}{2}$$

intersections. See Fig. 3.

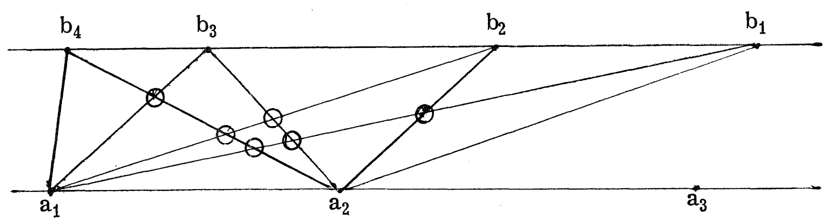

Fig. 3 $(p = 3, \ q = 4)$

The third set of q lines $(a_2, b_1), \cdots, (a_3, b_q)$ do not intersect with one another. The line (a_3, b_k) does intersect with the lines (a_1, b_j) and the lines (a_2, b_j) for $j = 1, 2, \cdots, (k - 1)$. Since here k may be equal to any of the integers $1, 2, \cdots,$ or q, the third set of lines contributes $2(1 + 2 + \cdots + q - 1)$ intersections. See Fig. 4.

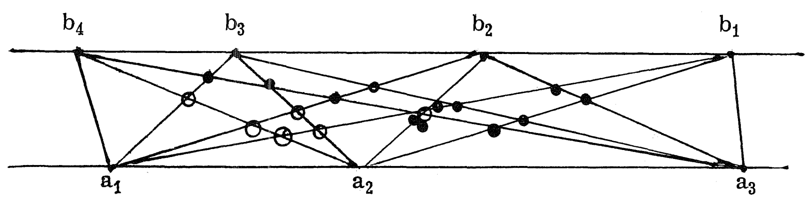

Fig. 4 $(p = 3, \ q = 4)$

Similarly, the r^{th} set of q lines $(a_r, b_1), \cdots, (a_r, b_q)$, $r = 1, 2, \cdots, p$ do not intersect with each other. The line (a_r, b_k) intersects with the lines (a_i, b_j), $i = 1, 2, \cdots, r - 1$ and $j = 1, 2, \cdots, (k - 1)$. Thus placement of the line (a_r, b_1) contributes no intersections, placement of (a_r, b_2) contributes $(r - 1)(1)$ intersections, placement of (a_r, b_3) contributes $(r - 1)(2)$ intersections, placement of (a_r, b_k) contributes $(r - 1)(k - 1)$ intersections and finally, placement of (a_r, b_q) contributes $(r - 1)(q - 1)$ intersections. In total, the r^{th} set of lines contributes $(r - 1)(1 + 2 + \cdots + q - 1)$ intersections.

Since the r^{th} set contributes $(r - 1)(1 + 2 + \cdots + q - 1)$ intersections and the index r may be $1, 2, \cdots,$ or p we have that the total number of intersections is

$$\sum_{r=1}^{p} (r - 1)(1 + 2 + \cdots + q - 1) = (1 + 2 + \cdots + p - 1)(1 + 2 + \cdots + q - 1),$$

which is, as shown in [1] by the same method of sequential line placement, also equal to

$$\frac{p(p-1)}{2}\,\frac{q(q-1)}{2} = \frac{1}{4}pq(p-1)(q-1).$$

Proof by G. Polya. In a private communication, Professor Polya has given the following shorter proof: Consider the trapezium of which the intersecting line segments are the diagonals. (See Fig. 5. The trapezium consists of (b_1, b_2), (b_3, a_2), (a_2, a_3), (a_3, b_1).)

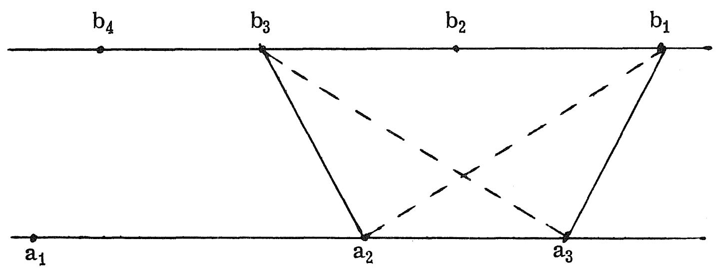

Fig. 5 $(p = 3, \; q = 4)$

Each trapezium is determined if a pair of points on each line is chosen and each different trapezium determines a different one of the intersections. Since there are

$$\binom{p}{2}\binom{q}{2} = \frac{p(p-1)}{2}\cdot\frac{q(q-1)}{2}$$

such choices, the result follows. This latter method of proof and the result are quite similar to the solution of the problem of finding the number of intersections of the diagonals of a convex polygon of n sides as discussed in [2].

REFERENCES

1. F. Stern, "The Sum of the First n Positive Integers — Geometrically," Fibonacci Quarterly, Vol. 9, No. 5, December, 1971, p. 526.
2. G. Polya, Mathematical Discovery, Vol. 1, Wiley, 1962, Problem 3.27, p. 83 and p. 178.