ADVANCED PROBLEMS AND SOLUTIONS

Edited by
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Send all communications concerning Advanced Problems and Solutions to Raymond E. Whitney, Mathematics Department, Lock Haven State College, Lock Haven, Pennsylvania 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, solutions should be submitted on separate signed sheets within two months after publication of the problems.

H-221 Proposed by L. Carlitz, Duke University, Durham, North Carolina

Let $p = 2m + 1$ be an odd prime, $p \neq 5$. Show that if m is even then

$$\begin{cases} F_m \equiv 0 \pmod{p} & \left(\left(\frac{5}{p}\right) = +1\right) \\ F_{m+1} \equiv 0 \pmod{p} & \left(\left(\frac{5}{p}\right) = -1\right) \end{cases};$$

if m is odd, then

$$\begin{cases} L_m \equiv 0 \pmod{p} & \left(\left(\frac{5}{p}\right) = +1\right) \\ L_{m+1} \equiv 0 \pmod{p} & \left(\left(\frac{5}{p}\right) = -1\right) \end{cases}$$

where $\left(\frac{5}{p}\right)$ is the Legendre symbol.

H-222 Proposed by R. E. Whitney, Lock Haven State College, Lock Haven, Pennsylvania.

A natural number, n, is called semiperfect, if there is a collection of distinct proper divisors of n whose sum is n. A number, n, is called abundant if $\sigma(n) > 2n$, where $\sigma(n)$ represents the sum of the distinct divisors of n (not necessarily proper). Finally a number, n, is called weird* if it is abundant and not semiperfect.

Are any Fibonacci or Lucas numbers weird? (All known weird numbers are even.)

*Elementary Problem E2308, American Mathematical Monthly, 79 (1972), p. 774.

H-223 Proposed by L. Carlitz and R. Scoville, Duke University, Durham, North Carolina.

Let S be a set of k elements. Find the number of sequences $(A_1, A_2, \cdots, A_n)$ where each A_i is a subset of S, and where $A_1 \subseteq A_2$, $A_2 \supseteq A_3$, $A_3 \subseteq A_4$, $A_4 \supseteq A_5$, etc.

H-224 Proposed by V. E. Hoggatt, Jr., San Jose State University, San Jose, California.

Let

$$A = \begin{vmatrix} 1 & 1 & 1 & 1 & \cdots \\ 1 & 2 & 3 & 4 & \cdots \\ 2 & 5 & 9 & 14 & \cdots \\ 3 & 10 & 22 & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix}_{n\times n}$$

denote the Fibonacci convolution determinant, and

$$B = \begin{vmatrix} 2 & 3 & 4 & \cdots & n+1 \\ 5 & 9 & 14 & \cdots & \\ 10 & 22 & \cdots & & \\ \cdots & & & & \end{vmatrix}_{n\times n},$$

where the first row and column of A have been deleted. Show

(i) $A = 1$ and

(ii) $B = n + 1$.

H-225 Proposed by Guy A. R. Guillotte, Quebec, Canada.

Let p denote an odd prime and $x^p + y^p = z^p$ for positive integers, x, y, and z. Show that for a large value of p,

$$p \doteq \frac{z}{z - x} + \frac{z}{z - y} .$$

Also show that

$$p < \frac{z}{z - x} + \frac{z}{z - y} .$$

H-226 Proposed by L. Carlitz and R. Scoville, Duke University, Durham, North Carolina.

(i) Let k be a fixed positive integer. Find the number of sequences of integers $(a_1, a_2, \cdots, a_n)$ such that

$$0 \le a_i \le k \quad (i = 1, 2, \cdots, n)$$

and if $a_i > 0$ then $a_i \ne a_{i-1}$ for $i = 2, \cdots, n$.

(ii) Let k be a fixed positive integer. Find the number of sequences of integers $(a_1, a_2, \cdots, a_n)$ such that

$$0 \le a_i \le k \quad (i = 1, 2, \cdots, n)$$

and if $a_i > 0$ then $a_i \ne a_{i-1}$ for $i = 2, \cdots, n$; moreover $a_i = 0$ for exactly r values of i.

SOLUTIONS

PROBABLY?

H-179 Proposed by D. Singmaster, Bedford College, University of London, England.

Let k numbers $p_1, p_2, \cdots, p_k$ be given. Set $a_n = 0$ for $n < 0$; $a_0 = 1$ and define a_n by the recursion

$$a_n = \sum_{i=1}^{n} p_i a_{n-i} \qquad n > 0 .$$

1. Find simple necessary and sufficient conditions on the p_i for $\lim_{n \to \infty} a_n$ to exist and be a) finite and non zero, b) zero, c) infinite.

2. Are the conditions: $p_i \geq 0$ for $i = 1, 2, \cdots, p_1 > 0$ and

$$\sum_{i=1}^{n} p_i = 1$$

sufficient for $\lim_{n \to \infty} a_n$ to exist, be finite and be nonzero?

Comment by the Proposer

This problem arises in the following probabilistic situation. We have a k-sided die (or other random device) such that the probability of i occurring is p_i. We have a game board consisting of a sequence of squares indexed 0, 1, 2, $\cdots$. Beginning at square 0, we use the die to determine the n number of squares moved, as in Monopoly or other board games. Then a_n gives the probability of landing on square n. Since the average (expected) move is

$$E = \sum_{i=1}^{n} i \cdot p_i ,$$

one would hope that $a_n \to 1/E$. If all $p_i = 1/k$, this can be seen.

The restriction $p_1 > 0$ (or some more complicated restriction) is necessary to avoid situations such as $p_1 = 0$, $p_2 = 1$ which gives $a_{2n} = 1$, $a_{2n+1} = 0$ for all n.

Editorial Comment

Since p_i would not be defined for $i > k$, an infinite set of numbers $\{p_1, p_2, \cdots\}$ would have to be given.