

## COUNTING OF CERTAIN PARTITIONS OF NUMBERS

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It can be shown from a recent discussion of subscript sets [1] that the counting of certain restricted, but useful, partitions can be made by means of binomial coefficients. In [1], these binomial coefficients were those which sum to a Fibonacci number.

A brief description of a subscript set is repeated here for continuity. A subscript set is defined as the set of all sequences of  $h$ , non-zero, non-repeating positive integers called subscripts, having the properties that no subscript exceeds  $M$ , no subscript is less than  $m$ , and that for all sequences one fixed parity order applies. Let it further be specified that each sequence be arranged in descending size order from the left. Under these conditions, there are unique starting and last sequences. The leftmost position of the starting sequence is occupied by a subscript  $k \leq M$  (depending on parity) with the other  $(h - 1)$  positions filled with the largest permissible subscripts. Correspondingly, the rightmost position of the last sequence is occupied by a subscript  $p \geq m$  with all other  $(h - 1)$  positions filled with the smallest subscripts possible.

A practical method for generating the sequences from the starting sequence is described in the reference [1] and, briefly, consists of progressive and exhaustive reduction of subscripts by two. The number of sequences in a subscript set is [1]

$$(1) \quad R_{h,q} = \binom{h + q/2}{q/2} ,$$

where  $q$  is the necessarily even difference between the rightmost subscripts of the starting and last sequences. If collections of subscript sets described as basic [1] are enumerated, the  $R_{h,q}$  for each set is one of a sum of binomial coefficients which sum to a Fibonacci number.

Suppose a new set of sequences called the q-set is formed whose sequences are formed by positionwise subtraction of the last sequence of a subscript set from all the others (including the last). The q-set sequences start with  $\underline{h}$   $q$ 's,  $(q, q, \dots, q)$  and end with  $\underline{h}$  zeros,  $(0, 0, \dots, 0)$  and are in one-to-one correspondence with the sequences of the subscript set. All the sequences of a q-set contain even numbers only.

Next, divide all integers of a q-set by two. It is seen that the set of sequences so produced are the  $h$  and less part partitions of  $(h q/2)$  with no integer exceeding  $q/2$ , plus the null partition  $(0, 0, \dots, 0)$  which can be conveniently discarded. Accordingly, in Chrystal's

notation\* for partition counting,

$$(2) \quad P(\leq hq/2 | \leq h | \leq q/2) = \binom{h + q/2}{q/2} - 1 ,$$

where the subtracted one accounts for the discarded null partition. A slightly different version of (2) states

$$(3) \quad P(\leq ab - b^2 | \leq a - b | \leq b) = \binom{a}{b} - 1 .$$

To each integer of the partitions counted by (2) or (3) including the null partition add one. The result is the  $h$ -part partitions of  $h(1 + q/2)$  with no member exceeding  $(1 + q/2)$ . Accordingly,

$$(4) \quad P(h + hq/2 | \leq h | \leq 1 + q/2) = \binom{h + q/2}{q/2} ,$$

and

$$(5) \quad P(a + ab - b - b^2 | \leq a - b | \leq b + 1) = \binom{a}{b} .$$

#### REFERENCES

1. D. C. Fielder, "A Discussion of Subscript Sets with Some Fibonacci Counting Help," submitted to the Fibonacci Quarterly.
2. G. Chrystal, Textbook of Algebra, Vol. 2 (Reprint), Chelsea, New York, 1952, p. 558.

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\*In Chrystal's notation  $P(A|B|C)$  represents a number of  $B$  part partitions of  $A$  with  $C$  the largest integer possible for any partition with no restrictions on  $A$ ,  $B$ , and  $C$  except as they occur naturally. Inequalities  $\leq$  affixed to some or all of the quantities specifies not greater than.

