COUNTING OMITTED VALUES

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1. INTRODUCTION

H. L. Alder in [1] has extended J. L. Brown, Jr.'s result on complete sequences [2] by showing that if $\{P_i\}_{i=1,2,\cdots}$ is a non-decreasing sequence and $\{k_i\}_{i=1,2,\cdots}$ is a sequence of positive integers, then with $P_1 = 1$ every natural number can be represented as

$$\sum_{i=1}^{\infty} \alpha_i P_i ,$$

where $0 \le \alpha_i \le k_i$ if and only if

$$P_{k+1} \le 1 + \sum_{i=1}^{n} k_i P_i$$

for $n = 1, 2, \cdots$. He also proves for a given sequence $\{k_i\}_{i=1,2,\cdots}$ there is only one non-decreasing sequence of positive integers $\{P_i\}_{i=1,2,\cdots}$ for which the representation is unique for every natural number, namely the set $\{P_i\} = \{\phi_i\}_{i=1,2,\cdots}$ where $\phi_1 = 1$, $\phi_2 = 1 + k_1$, $\phi_3 = (1 + k_1)(1 + k_2)$, $\cdots$, $\phi_i = (1 + k_1)(1 + k_2) \cdots (1 + k_{i-1})$, $\cdots$.

This paper investigates those natural numbers not represented by the form

$$\sum_{i=1}^{\infty} \alpha_i P_i$$

for $0 \le \alpha_i \le k_i$ where $\{k_i\}$ is as above and $\{P_i\}_{i=1,2,\cdots}$ is a necessarily increasing sequence of positive integers satisfying

$$(1) \qquad P_{n+1} \ge 1 + \sum_{i=1}^{n} k_i P_i \qquad n = 1, 2, \cdots .$$

When specialized to $k_i = 1$ for all i the results obtained include those in Hoggatt's and Peterson's paper [3].

2. UNIQUENESS OF REPRESENTATION

Theorem 1. For P_i satisfying (1), the representation of the natural number N as

$$\sum_{i=1}^{\infty} \alpha_i P_i ,$$

where $0 \le \alpha_i \le k_i$ is unique.

Proof. Let N be the smallest integer with possibly two representations

$$N = \sum_{s=1}^{n} \alpha_s P_s = \sum_{t=1}^{n} \beta_t P_t ,$$

where $\alpha_n \ne 0 \ne \beta_m$.

If $m \ne n$ assume $m > n$. Then by (1)

$$\sum_{s=1}^{n} \alpha_s P_s \le \sum_{s=1}^{n} k_s P_s \le P_{n+1} - 1 \le P_m - 1 \le \sum_{t=1}^{m} \beta_t P_t - 1 < \sum_{t=1}^{m} \beta_t P_t .$$

Thus, $m = n$. Either $\alpha_n \ge \beta_n$ or $\alpha_n \le \beta_n$. Suppose without loss of generality $\alpha_n \ge \beta_n$. The natural number

$$\sum_{t=1}^{n-1} \beta_t P_t = \sum_{s=1}^{n-1} \alpha_s P_s + (\alpha_n - \beta_n) P_n$$

and since it is less than N it has only one representation. Hence $\alpha_s = \beta_s$ for $s = 1, 2, \cdots, n$, i.e., N has a unique representation.

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Definition. For $x \ge 0$ let $M(x)$ be the number of natural numbers less than or equal to x which are not represented by

$$\sum_{i=1}^{\infty} \alpha_i P_i .$$

Theorem 2. If

$$N = \sum_{i=1}^{n} \alpha_i P_i, \qquad \alpha_n \neq 0 ,$$

is the largest representable integer not exceeding the positive number x then

$$M(x) = [x] - \sum_{i=1}^{n} \alpha_i \phi_i ,$$

where [ ] is the greatest integer function.

Proof. By Theorem 1, it is sufficient to show the number, $R(x)$, of representable integers not exceeding x, equals

$$\sum_{i=1}^{n} \alpha_i \phi_i .$$

But $R(x) = R(N)$ from the definition of N. Now all integers of the form

$$\sum_{i=1}^{n} \beta_i P_i$$

with the only restriction that $0 \leq \beta_n < \alpha_n$ are less than N since:

$$\begin{aligned} \sum_{i=1}^{n} \beta_i P_i &\leq \sum_{i=1}^{n-1} k_i P_i + \beta_n P_n \\ &\leq \{1 + \beta_n\} P_n - 1 \\ &< \alpha_n P_n + \sum_{i=1}^{n-1} \alpha_i P_i = N . \end{aligned}$$

Again by the uniqueness of representation to form

$$\sum_{i=1}^{n} \beta_i P_i, \qquad 0 \le \beta_n < \alpha_n ,$$

there are α_n choices for β_n, $\{1 + k_{n-1}\}$ choices for β_{n-1}, $\{1 + k_{n-2}\}$ choices for β_{n-2}, $\cdots$, and $\{1 + k_1\}$ choices for β_1 ; in all there are $\alpha_n \phi_n$ numbers.

It remains to count numbers of the form

$$\alpha_n P_n + \sum_{i=1}^{n-1} \beta_i P_i$$

which do not exceed

$$N = \alpha_n P_n + \sum_{i=1}^{n-1} \alpha_i P_i .$$

That is the number of integers

$$\sum_{i=1}^{n-1} \beta_i P_i \le \sum_{i=1}^{n-1} \alpha_i P_i .$$

Hence

$$R\left(\sum_{i=1}^{n} \alpha_i P_i\right) = \alpha_n \phi_n + R\left(\sum_{i=1}^{n-1} \alpha_i P_i\right)$$

and because $R\{\alpha_1 P_1\} = \alpha_1 = \alpha_1 \phi_1$ then

$$R\left(\sum_{i=1}^{n} \alpha_i P_i\right) = \sum_{i=1}^{n} \alpha_i \phi_i .$$

[The representable positive integers less than or equal to $\alpha_1 P_1$ are P_1, $2P_1$, $\cdots$, $\alpha_1 P_1$.] This completes the proof.

As P_i is representable, the theorem give $M(P_i) = P_i - \phi_i$ and the following result is immediate.

Corollary.

$$M\left(\sum_{i=1}^{n} \alpha_i P_i\right) = \sum_{i=1}^{n} \alpha_i M(P_i) = \sum_{i=1}^{n} M(\alpha_i P_i) .$$

Note that if $k_i = 1$ for all $i = 1, 2, \cdots$, Theorems 3 and 4 in [3] are special cases of the above theorem and corollary.

4. SOME APPLICATIONS

The two sequences $P_n = F_{2n}$ and $P_n = F_{2n-1}$, $n = 1, 2, \cdots$ mentioned in [3] satisfy Theorems 1 and 2 for $k_i = 1$ $i = 1, 2, \cdots$. However,

$$1 + 2(F_2 + F_4 + \cdots + F_{2n}) = F_{2n+2} + F_{2n-1} - 1 \geq F_{2n+2}$$

with equality only when $n = 1$, and

$$1 + 2(F_1 + \cdots + F_{2n-1}) = F_{2n+1} + F_{2n-2} + 1 > F_{2n+1} .$$

Consequently, by Alder's result,

Theorem 3. Every natural number can be expressed as

$$\sum_{i=1}^{\infty} \alpha_i F_{2i}$$

and as

$$\sum_{i=1}^{\infty} \beta_i F_{2i-1} ,$$

where α_i and β_i are 0, 1, or 2.

To return to the general case, let $\{k_i\}$ be a fixed sequence of positive integers; then any sequence $\{P_i\}$ satisfying (1) also satisfies $P_n \geq \phi_n$ for all n. This follows from $P_1 \geq 1 = \phi_1$ and induction:

$$P_n \geq 1 + \sum_{i=1}^{n-1} k_i P_i \geq 1 + \sum_{i=1}^{n-1} k_i \phi_i = \phi_n .$$

Hence by the corollary

$$M\left(\sum_{i=1}^{n} \alpha_i P_i\right) = \sum_{i=1}^{n} \alpha_i \{P_i - \phi_i\} \geq 0 ,$$

with equality iff $P_i = \phi_i$. Furthermore, since $k_i \geq 1$ then $\{\phi_i\}$ is an increasing sequence and so for every natural number N there exist α_i such that

$$N < \sum_{i=1}^{\infty} \alpha_i \phi_i .$$

Therefore

$$M(N) \leq M\left(\sum_{i=1}^{\infty} \alpha_i \phi_i\right) = 0 ,$$

i.e., N has a representation in the form

$$\sum_{i=1}^{\infty} \beta_i \phi_i .$$

These facts, together with Theorem 1, give

Theorem 4. If $\{k_i\}$ is any sequence of positive integers, then every natural number has a unique representation as

$$\sum_{i=1}^{\infty} \alpha_i \phi_i ,$$

where $0 \leq \alpha_i \leq k_i$ and $\phi_1 = 1$, $\phi_2 = (1 + k_1)$, $\cdots$, $\phi_i = (1 + k_1) \cdots (1 + k_{i-1})$.

Corollary. If r is a fixed integer larger than 1 then every natural number has a unique representation in base r.

Proof. In Theorem 4, take $1 + k_i = r$ for all i.

5. REFERENCES

1. H. L. Alder, "The Number System in More General Scales," Mathematics Magazine, Vol. 35 (1962), pp. 145-151.
2. John L. Brown, Jr., "Note on Complete Sequences of Integers," The American Math. Monthly, Vol. 67 (1960), pp. 557-560.
3. V. E. Hoggatt, Jr., and Brian Peterson, "Some General Results on Representations," Fibonacci Quarterly, Vol. 10 (1972), pp. 81-88.