A PRIMER FOR THE FIBONACCI NUMBERS: PART XIII

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THE FIBONACCI CONVOLUTION TRIANGLE, PASCAL'S TRIANGLE, AND SOME INTERESTING DETERMINANTS

The simplest and most well-known convolution triangle is Pascal's triangle, which is formed by convolving the sequence $\{1, 1, 1, \cdots\}$ with itself repeatedly. The Fibonacci convolution triangle [1] is formed by repeated convolutions of the sequence $\{1, 1, 2, 3, 5, 8, 13, \cdots\}$ with itself. We now show three different ways to obtain the Fibonacci convolution triangle, as well as some interesting sequences of determinant values found in Pascal's triangle, the Fibonacci convolution triangle, and the trinomial coefficient triangle.

1. CONVOLUTION OF SEQUENCES

If $\{a_n\}$ and $\{b_n\}$ are two sequences, then the convolution of the two sequences is another sequence $\{c_n\}$ which is calculated as shown:

$$
\begin{aligned}
c_1 &= a_1b_1 \\
c_2 &= a_1b_2 + a_2b_1 \\
c_3 &= a_1b_3 + a_2b_2 + a_3b_1 \\
&\cdots\cdots\cdots\cdots \\
c_n &= a_1b_n + a_2b_{n-1} + a_3b_{n-2} + \cdots + a_nb_1 = \sum_{k=1}^{n} a_kb_{n-k+1} \,.
\end{aligned}
$$

If we convolve the Fibonacci sequence with itself, we obtain the First Fibonacci Convolution Sequence $\{1, 2, 5, 10, 20, 38, 71, \cdots\}$, as follows:

$$
\begin{aligned}
F_1^{(1)} &= F_1F_1 && = 1\cdot1 && = 1 \\
F_2^{(1)} &= F_1F_2 + F_2F_1 && = 1\cdot1 + 1\cdot1 && = 2 \\
F_3^{(1)} &= F_1F_3 + F_2F_2 + F_3F_1 && = 1\cdot2 + 1\cdot1 + 2\cdot1 && = 5 \\
F_4^{(1)} &= F_1F_4 + F_2F_3 + F_3F_2 + F_4F_1 && = 1\cdot3 + 1\cdot2 + 2\cdot1 + 3\cdot1 && = 10 \\
&\cdots\cdots\cdots\cdots\cdots\cdots
\end{aligned}
$$

Next we can obtain the Second Fibonacci Convolution Sequence $\{1, 3, 9, 22, 51, 111, \cdots\}$ as indicated below.

$$\begin{array}{lllll}
F_1^{(2)} & = F_1F_1^{(1)} & = 1\cdot 1 & = 1 \\
F_2^{(2)} & = F_2F_1^{(1)} + F_1F_2^{(1)} & = 1\cdot 1 + 1\cdot 2 & = 3 \\
F_3^{(2)} & = F_3F_1^{(1)} + F_2F_2^{(1)} + F_1F_3^{(1)} & = 2\cdot 1 + 1\cdot 2 + 1\cdot 5 & = 9 \\
F_4^{(4)} & = F_4F_1^{(1)} + F_3F_2^{(1)} + F_2F_3^{(1)} + F_1F_4^{(1)} & = 3\cdot 1 + 2\cdot 2 + 1\cdot 5 + 1\cdot 10 & = 22 \\
\cdots & \cdots & \cdots & \cdots
\end{array}$$

by writing the convolution of the first Fibonacci convolution sequence with the Fibonacci sequence. To obtain the succeeding Fibonacci convolution sequences, we continue writing the convolution of a Fibonacci convolution sequence with the Fibonacci sequence. A second method follows.

The Fibonacci sequence is obtained from the generating function

$$\frac{1}{1 - x - x^2} = F_1 + F_2x + F_3x^2 + \cdots + F_{n+1}x^n + \cdots ,$$

which provides Fibonacci numbers as coefficients of successive powers of x as far as one pleases to carry out a long division. The k^{th} convolution of the Fibonacci numbers appears as the coefficients of successive powers of x in the generating function

$$\frac{1}{(1 - x - x^2)^{k+1}} = F_1^{(k)} + F_2^{(k)}x + F_3^{(k)}x^2 + \cdots + F_{n+1}^{(k)}x^n + \cdots ,$$

$k = 0, 1, 2, \cdots$. For $k = 0$, we get just the Fibonacci numbers. In the next section, we shall see yet another way to find the convolved Fibonacci sequences.

3. THE FIBONACCI CONVOLUTION TRIANGLE

Suppose someone writes a column of zeroes. To the right and one space down place a one. To generate the elements below the one we add the one element directly above and the one element diagonally left of the element to be written. Such a rule generates a convolution triangle. This rule, of course, generates Pascal's triangle in left-justified form:

$$\begin{array}{ccccccc}
0 \\
0 & 1 \\
0 & 1 & 1 \\
0 & 1 & 2 & 1 \\
0 & 1 & 3 & 3 & 1 \\
0 & 1 & 4 & 6 & 4 & 1 \\
0 & 1 & 5 & 10 & 10 & 5 & 1 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots
\end{array}$$

The columns of Pascal's triangle give convolution sequences for the sequence $\{1, 1, 1, \cdots\}$. Notice that the row sums give powers of two, and the sums of rising diagonals formed by beginning in the column of ones and going up one and to the right one throughout the array give the Fibonacci numbers $1, 1, 2, 3, 5, \cdots, F_n, \cdots$, where $F_n = F_{n-1} + F_{n-2}$, $n = 3, 4, 5, \cdots$.

Next suppose we change the rule of formation. Begin as before, but to generate elements below the one, add the two elements directly above and the element diagonally left of the element to be generated. Now we have the Fibonacci convolution triangle in left-justified form,

$$\begin{array}{ccccccc}
0 \\
0 & 1 \\
0 & 1 & 1 \\
0 & 2 & 2 & 1 \\
0 & 3 & 5 & 3 & 1 \\
0 & 5 & 10 & 9 & 4 & 1 \\
0 & 8 & 20 & 22 & 14 & 5 & 1 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{array}$$

The columns give the convolution sequences for the Fibonacci sequence. The row sums are the Pell numbers $1, 2, 5, 12, 29, 70, \cdots, p_n, \cdots$, where $p_n = 2p_{n-1} + p_{n-2}$. The rising diagonal sums are $1, 1, 3, 5, 11, 21, \cdots, r_n, \cdots$, where $r_n = r_{n-1} + 2r_{n-2}$. The diagonal sums found by beginning in the column of Fibonacci numbers and going up two and right one throughout the array are $1, 1, 2, 4, 7, 13, 24, \cdots, T_n, \cdots, T_n = T_{n-1} + T_{n-2} + T_{n-3}$, the Tribonacci numbers.

If one changes the rule of formation yet again, so that the elements below the initial one are found by adding the one element directly above and the two elements diagonally left of the element to be generated, the array obtained is the trinomial coefficient triangle. The coefficients in successive rows are the same as those found in expansions of the trinomial $(1 + x + x^2)^n$, $n = 0, 1, 2, \cdots$. The columns do not form convolution sequences as before, but the row sums are now the powers of three, and the sums of elements appearing on the rising diagonals are $1, 1, 2, 4, 7, 13, \cdots$, the Tribonacci numbers just defined. To illustrate, the trinomial triangle is formed as follows:

$$\begin{array}{cccccccccc}
0 \\
0 & 1 \\
0 & 1 & 1 & 1 \\
0 & 1 & 2 & 3 & 2 & 1 \\
0 & 1 & 3 & 6 & 7 & 6 & 3 & 1 \\
0 & 1 & 4 & 10 & 16 & 19 & 16 & 10 & 4 & 1 \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{array}$$

3. SOME SPECIAL MATRICES

If one looks again at how convoluted sequences are formed, the arithmetic is much like matrix multiplication. Suppose that we define three matrices. Let P be the $n \times n$ matrix formed by using as elements the first n rows of Pascal's triangle in rectangular form. Let F be the $n \times n$ matrix formed by writing the first n rows of Pascal's triangle in vertical position on and below the main diagonal, which makes the row sums of F be Fibonacci numbers. Let C be the $n \times n$ matrix whose elements are the first n rows of the Fibonacci convolution triangle written in rectangular form. Then it can be proved that FP = C (see [1], [2].) To illustrate, for n = 6,

$$
(3.1)\qquad \begin{aligned}
FP &= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 3 & 1 & 0 \\ 0 & 0 & 0 & 3 & 4 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 3 & 6 & 10 & 15 & 21 \\ 1 & 4 & 10 & 20 & 35 & 56 \\ 1 & 5 & 15 & 35 & 70 & 126 \\ 1 & 6 & 21 & 56 & 126 & 252 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 5 & 9 & 14 & 20 & 27 \\ 3 & 10 & 22 & 40 & 65 & 98 \\ 5 & 20 & 51 & 105 & 190 & 315 \\ 8 & 38 & 111 & 256 & 511 & 924 \end{bmatrix} = C
\end{aligned}
$$

Suppose that, instead of multiplying matrix F by the rectangular Pascal array P, we use an $n \times n$ matrix A whose elements are given by the first n rows of Pascal's triangle in left-justified form on and below its main diagonal, and zero elsewhere. Let F^t be the transpose of F. Then the matrix product $AF^t = T$, where T is the $n \times n$ matrix whose elements are found in the left-justified trinomial coefficient triangle given in Section 2. We illustrate for n = 6:

$$
(3.2)\qquad \begin{aligned}
AF^t &= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 \\ 1 & 3 & 3 & 1 & 0 & 0 \\ 1 & 4 & 6 & 4 & 1 & 0 \\ 1 & 5 & 10 & 10 & 5 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 1 & 3 & 3 \\ 0 & 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 2 & 3 & 2 & 1 & 0 \\ 1 & 3 & 6 & 7 & 6 & 3 \\ 1 & 4 & 10 & 16 & 19 & 16 \\ 1 & 5 & 15 & 30 & 45 & 51 \end{bmatrix} = T \; .
\end{aligned}
$$

4. SPECIAL DETERMINANTS IN PASCAL'S TRIANGLE

A multitude of unit determinants can be found in Pascal's triangle. The following theorems are proved in [2].

Theorem 4.1. The determinant of any $k \times k$ array taken with its first column along the column of ones and its first row the i^{th} row of Pascal's triangle written in left-justified form, has value one.

Theorem 4.2. The determinant of any $k \times k$ array taken with its first row along the row of ones or with its first column along the column of ones in Pascal's triangle written in rectangular form, is one.

For example,

$$1 = \begin{vmatrix} 1 & 2 & 1 & 0 \\ 1 & 3 & 3 & 1 \\ 1 & 4 & 6 & 4 \\ 1 & 5 & 10 & 10 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 4 & 5 & 6 \\ 10 & 15 & 21 \end{vmatrix} = \begin{vmatrix} 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & 20 \\ 1 & 5 & 15 & 35 \\ 1 & 6 & 21 & 56 \end{vmatrix} .$$

Pascal's triangle also has sequences of determinants which have binomial coefficients for their values. Here we have to number the rows and columns of Pascal's triangle; the row of ones is the zero^{th} row; the column of ones the zero^{th} column. To illustrate some of the sequences of determinants considered here, we look back at the matrix P of (3.1) which contains the first n rows and columns of Pascal's triangle written in rectangular form. When 2×2 determinants are taken across the first and second rows of Pascal's rectangular array,

$$\begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix} = 1, \quad \begin{vmatrix} 2 & 3 \\ 3 & 6 \end{vmatrix} = 3, \quad \begin{vmatrix} 3 & 4 \\ 6 & 10 \end{vmatrix} = 6, \quad \begin{vmatrix} 4 & 5 \\ 10 & 15 \end{vmatrix} = 10, \quad \cdots,$$

giving values found in the second column of Pascal's triangle. Of course, the 1×1 determinants along the first row give the values found in the first column of Pascal's triangle. Taking 3×3 determinants yields

$$\begin{vmatrix} 1 & 2 & 3 \\ 1 & 3 & 6 \\ 1 & 4 & 10 \end{vmatrix} = 1, \quad \begin{vmatrix} 2 & 3 & 4 \\ 3 & 6 & 10 \\ 4 & 10 & 20 \end{vmatrix} = 4, \quad \begin{vmatrix} 3 & 4 & 5 \\ 6 & 10 & 15 \\ 10 & 20 & 35 \end{vmatrix} = 10, \quad \cdots,$$

successive entries in the third column of Pascal's triangle. In fact, taking successive $k \times k$ determinants along the first, second, $\cdots$, and k^{th} rows yields the successive entries of the k^{th} column of Pascal's triangle.

The following theorems are proved in [3].

Theorem 4.3. If Pascal's triangle is written in left-justified form, any $k \times k$ matrix selected within the array with its first column the first column of Pascal's triangle and its first row the i^{th} row has determinant value given by the binomial coefficient

$$\binom{i+k-1}{k} .$$

Theorem 4.4. The determinant of the $k \times k$ matrix taken with its first column the j^{th} column of Pascal's triangle written in rectangular form, and its first row the first row of the rectangular Pascal array, has values given by the binomial coefficient

$$\binom{j+k-1}{k} .$$

5. SPECIAL DETERMINANTS IN THE FIBONACCI CONVOLUTION TRIANGLE AND IN THE TRINOMIAL TRIANGLE ARRAYS

Now we are ready to prove that the unit determinants and binomial coefficient determinants of Section 4 are also found in the Fibonacci convolution triangle and in the trinomial coefficient triangle. Returning to (3.1), the first n entries of the first n rows of the Fibonacci convolution triangle are given by the matrix product $FP = C$. But, notice that $k \times k$ submatrices of C taken along either the first or second matrix row are the product of a $k \times k$ submatrix of F with a unit determinant and a similarly placed $k \times k$ submatrix of P which has been evaluated in Theorem 4.2 or Theorem 4.4. Let us also number the Fibonacci convolution triangle as Pascal's triangle, with the top row the zeroth row. Thus, we have

<u>Theorem 5.1</u>. Let a $k \times k$ matrix M be selected from the Fibonacci convolution triangle in rectangular form. If M includes the row of ones, then $\det M = 1$. If M has its first column the j^{th} column and its first row along the first row of the Fibonacci array, then

$$\det M = \binom{j+k-1}{k} .$$

Reasoning in a similar fashion from (3.2), the matrix product AF^t and Theorems 4.1 and 4.3 yield the following, where the trinomial coefficient triangle is numbered as Pascal's triangle, with the left-most column the zeroth column.

<u>Theorem 5.2.</u> Let a $k \times k$ matrix N be selected from the trinomial triangle written in left-justified form. If N includes the column of ones, then $\det N = 1$. If N has its first row the i^{th} row and its first column along the first column of the trinomial triangle, then

$$\det N = \binom{i+k-1}{k} .$$

These results are generalized in [2] and [3]. Other classes of determinants are also developed there. The reader should verify the results given here numerically.

REFERENCES

1. Verner E. Hoggatt, Jr., "Convolution Triangles for Generalized Fibonacci Numbers," <u>Fibonacci Quarterly</u>, Vol. 8, No. 2, March 1970, pp. 158-171.
2. Marjorie Bicknell and V. E. Hoggatt, Jr., "Unit Determinants in Generalized Pascal Triangles," <u>Fibonacci Quarterly</u>, Vol. 11, No. 2, April, 1973, pp. 131-144.
3. Marjorie Bicknell and V. E. Hoggatt, Jr., "Special Determinants Found within Generalized Pascal Triangles," <u>Fibonacci Quarterly</u>, Vol. 11, No. 5, Dec., 1973, pp. 469-479.