Send all communications regarding Elementary Problems and Solutions to Professor A. P. Hillman, Dept. of Mathematics and Statistics, University of New Mexico, Albuquerque, New Mexico 87131. Each problem or solution should be submitted in legible form, preferably typed in double spacing, on a separate sheet or sheets, in the format used below. Solutions should be received within four months of the publication date.

**Definitions.** The Fibonacci numbers \( F_n \) and the Lucas numbers \( L_n \) satisfy \( F_{n+2} = F_{n+1} + F_n \), \( F_0 = 0, F_1 = 1 \), and \( L_{n+2} = L_{n+1} + L_n \), \( L_0 = 2, L_1 = 1 \).

**Problems Proposed in This Issue**

**B-268** Proposed by Warren Cheves, Littleton, North Carolina.

Define a sequence of complex numbers \( \{ C_n \}, n = 1, 2, \ldots \), where \( C_n = F_n + iF_{n+1} \).

Let the conjugate of \( C_n \) be \( \overline{C}_n = F_n - iF_{n+1} \).

Prove

(a) \( C_n \overline{C}_n = F_{2n+1} \)

(b) \( C_n \overline{C}_{n+1} = F_{2n+2} + (-1)^n \).

**B-269** Proposed by Warren Cheves, Littleton, North Carolina.

Let \( Q \) be the matrix

\[
\begin{pmatrix}
1 & 1 \\
1 & 0
\end{pmatrix}
\]

The eigenvalues of \( Q \) are \( \alpha \) and \( \beta \), where \( \alpha = (1 + \sqrt{5})/2 \) and \( \beta = (1 - \sqrt{5})/2 \). Since the eigenvalues of \( Q \) are distinct, we know that \( Q \) is similar to a diagonal matrix \( A \). Show that \( A \) is either

\[
\begin{pmatrix}
\alpha & 0 \\
0 & \beta
\end{pmatrix}
\] or \[
\begin{pmatrix}
\beta & 0 \\
0 & \alpha
\end{pmatrix}
\].

**B-270** Proposed by Herta T. Freitag, Roanoke, Virginia.

Establish or refute the following: If \( k \) is odd,

\[ L_k \mid [F_{(n+2)k} - F_{nk}] \]

**B-271** Proposed by Herta T. Freitag, Roanoke, Virginia.

Establish or refute the following: If \( k \) is even, \( L_k - 2 \) is an exact divisor of

\( F_{(n+2)k} + 2F_k - F_{nk} \).
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(b) \( F_{(n+2)k} - 2F_{(n+1)k} = F_{nk} \); and
(c) \( 2\left[ F_{(n+2)k} - F_{(n+1)k} + F_{nk} \right] \).

B-272 Proposed by Gary G. Ford, Vancouver, British Columbia, Canada.

Find at least some of the sequences \( \{ y_n \} \) satisfying

\[ y_{n+3} + y_n = y_{n+2}y_{n+1} \]

B-273 Proposed by Marjorie Bicknell, A. C. Wilcox High School, Santa Clara, California.

Construct any triangle \( \triangle ABC \) with vertex angle \( A = 54^\circ \) and median \( AM \) to the side opposite \( A \) such that \( AM = 1 \). Now, inscribe \( \triangle XYM \) in \( \triangle ABC \) so that \( M \) is the midpoint of \( BC \), and \( X \) and \( Y \) lie between \( A \) and \( B \) and between \( A \) and \( C \), respectively. Find the minimum perimeter possible for the inscribed triangle, \( \triangle XYM \).

SOLUTIONS

POLYNOMIALS IN THE Q MATRIX

B-244 Proposed by J. L. Hunsucker, University of Georgia, Athens, Georgia.

Let \( Q \) be the \( 2 \times 2 \) matrix

\[
\begin{pmatrix}
1 & 1 \\
1 & 0
\end{pmatrix}
\]

and let

\[
M = \begin{pmatrix}
a & b \\
c & d
\end{pmatrix}
\]

be the sum of a finite number of matrices chosen from the sequence \( Q, Q^2, Q^3, \ldots \). Prove that \( b = c \) and \( a = b + d \).


It will be sufficient to show for \( n = 1, 2, 3, \ldots \) that if

\[
Q^n = \begin{pmatrix}
a & b \\
c & d
\end{pmatrix}
\]

then \( b = c \) and \( a = b + d \). For each \( Q^n \) has this property then the sum of a finite number of terms from the sequence \( Q, Q^2, Q^3, \ldots \) will retain the same property.

However, if

\[
Q = \begin{pmatrix}
1 & 1 \\
1 & 0
\end{pmatrix}
\]

then it is easily shown by induction that

\[
Q^n = \begin{pmatrix}
F_{n+1} & F_n \\
F_n & F_{n-1}
\end{pmatrix}
\]

for \( n \geq 1 \), and clearly this latter matrix has the required property.

ELEMENTARY PROBLEMS AND SOLUTIONS

SUMS AND DIFFERENCES OF FIBONACCI SQUARES

B-245 Proposed by Richard M. Grassl, University of New Mexico, Albuquerque, New Mexico.

Show that each term $F_n$ with $n > 0$ in the sequence $F_0$, $F_1$, $F_2$, $\ldots$ is expressible as $x^2 + y^2$ or $x^2 - y^2$ with $x$ and $y$ terms of the sequence with distinct subscripts.

Solution by David Zeitlin, Minneapolis, Minnesota.

The result follows by noting that $F_{2n} = F^2_{n+1} - F^2_{n-1}$ and $F_{2n-1} = F^2_n + F^2_{n-1}$.


AT MOST ONE IS RATIONAL

B-246 Proposed by L. Carlitz, Duke University, Durham, North Carolina.

Show that at least one of the following sums is irrational.

$$\sum_{n=0}^{\infty} \frac{1}{F_{2n+1}}, \quad \sum_{n=0}^{\infty} \frac{(-1)^n}{F_{2n+1}}.$$


Since (FQ, Vol. 5, pp. 469-471) sum I is $\sqrt{5}$ times sum II, sum I is irrational if sum II is rational, completing the proof.

Also solved by Paul S. Bruckman and the Proposer.

LUCAS MULTIPLES OF FIBONACCI NUMBERS

B-247 Proposed by Larry Lang, Student, San Jose State University, San Jose, California.

Given that $m$ and $n$ are integers with $0 < n < m$ and $F_n \text{L}_m$, prove that $n$ is 1, 2, 3, or 4.

Solution by Phil Mena, University of New Mexico, Albuquerque, New Mexico.

Let $m = qn + r$ with $m$, $n$, and $q$ positive integers and $0 \leq r < n$. Since $\gcd(F_n, F_{n+1}) = 1$ and $L_m = L_{m-n}F_{n+1} + L_{n-n-1}F_n$,

$F_n \text{L}_m$ implies $F_n \text{L}_{m-n}$. Continuing this way, one shows that $F_n \text{L}_m$ implies $F_n \text{L}_{m-qn}$, i.e., $F_n \text{L}_r$. Then $F_n < L_r$, $r < n$, and $n > 4$ imply $r = n - 1$ since it is easily shown by induction that $F_n > L_r$ for $n > 4$ and $r < n - 1$. Since $L_{n-1} = F_n + F_{n-2}$, $F_n \text{L}_{n-1}$ implies $F_n \text{L}_{n-2}$. This is impossible for $n > 2$, completing the proof.
SOME CASES OF $n \mid F_n$

B-248 Proposed by V. E. Hoggatt, Jr., San Jose State University, San Jose, California.

Let $k$ be a positive integer and let $h = 5^k$. Prove that $h \mid F_h$.


Proof by induction:

Let $h \mid F_h$ for $k = n$, and note that for $n = 1$, $h = 5 \mid F_5 = 5$. The factorization $x^5 - y^5 = (x - y)(x^4 + x^3y + x^2y^2 + xy^3 + y^4)$ with $x = \alpha^h$ and $y = \beta^h$ yields

$$F_{5h} = F_h (L_{4h} - L_{2h} + 1).$$

But $L_{4h} - L_{2h} + 1 = (5 F_{2h}^2 + 2) - (5 F_{h}^2 - 2) + 1 \equiv 0 \pmod{5}$. (Proof by induction.) Hence $F_{5h}$ is divisible by 5 if $F_h$ is divisible by n, which completes the induction.


EXAMPLES OF $n \mid L_n$

B-249 Proposed by V. E. Hoggatt, Jr., San Jose State University, San Jose, California.

Let $k$ be a positive integer and let $g = 2 \cdot 3^k$. Prove that $g \mid L_g$.


It will be shown that if $k$ is a positive integer and $g = 2 \cdot 3^k$ then $3g \mid L_g$ but $(9g) \not\mid L_g$, which implies the property asked in B–249.

Proof by induction.

Let the induction hypothesis be for $k = n$, $3g \mid L_g$ but $(9g) \not\mid L_g$. For $n = 1$ the hypothesis is true since $3g = 18 = L_4$. From the induction hypothesis $L_g = 3gt$, where $3$ and $t$ are coprime. Then

$$L_{3g} = L_g (L_{2g} - 1) \quad \text{[from } x^3 + y^3 = (x + y)(x^2 + xy + y^2)\text{]}$$

$$= 3gt(L_{2g} - 3) \quad (L_4, \text{ p. 59, of Hoggatt's book})$$

$$= 9gt(3g^2 t^2 - 1),$$

which shows that $3(3g) \mid L_{3g}$ but $(9g) \not\mid L_{3g}$.

Also solved by Paul S. Bruckman, Warren Cheves, Herta T. Freitag, Gregory Wulczyn, David Zeitlin, and the Proposer.