ELEMENTARY PROBLEMS AND SOLUTIONS

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<u>Definitions</u>. The Fibonacci numbers F_n and the Lucas numbers L_n satisfy $F_{n+2} = F_{n+1} + F_n$, $F_0 = 0$, $F_1 = 1$, and $L_{n+2} = L_{n+1} + L_n$, $L_0 = 2$, $L_1 = 1$.

PROBLEMS PROPOSED IN THIS ISSUE

B-268 Proposed by Warren Cheves, Littleton, North Carolina.

Define a sequence of complex numbers $\{C_n\}$, $n = 1, 2, \cdots$, where $C_n = F_n + iF_{n+1}$. Let the conjugate of C_n be $\overline{C}_n = F_n - iF_{n+1}$. Prove

(a) $C_n\overline{C}_n = F_{2n+1}$

(b) $C_n\overline{C}_{n+1} = F_{2n+2} + (-1)^n i$.

B-269 Proposed by Warren Cheves, Littleton, North Carolina.

Let Q be the matrix

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}.$$

The eigenvalues of Q are α and β, where $\alpha = (1 + \sqrt{5})/2$ and $\beta = (1 - \sqrt{5})/2$. Since the eigenvalues of Q are distinct, we know that Q is similar to a diagonal matrix A. Show that A is either

$$\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} \beta & 0 \\ 0 & \alpha \end{pmatrix}.$$

B-270 Proposed by Herta T. Freitag, Roanoke, Virginia.

Establish or refute the following: If k is odd,

$$L_k \mid [F_{(n+2)k} - F_{nk}] .$$

B-271 Proposed by Herta T. Freitag, Roanoke, Virginia.

Establish or refutue the following: If k is even, $L_k - 2$ is an exact divisor of

(a) $$F_{(n+2)k} + 2F_k - F_{nk} ;$$

(b) $F_{(n+2)k} - 2F_{(n+1)k} + F_{nk}$; and

(c) $2\left[F_{(n+2)k} - F_{(n+1)k} + F_k\right]$.

B-272 Proposed by Gary G. Ford, Vancouver, British Columbia, Canada.

Find at least some of the sequences $\{y_n\}$ satisfying

$$y_{n+3} + y_n = y_{n+2}\,y_{n+1} \, .$$

B-273 Proposed by Marjorie Bicknell, A. C. Wilcox High School, Santa Clara, California.

Construct any triangle ΔABC with vertex angle $A = 54^\circ$ and median $\overline{AM}$ to the side opposite A such that $AM = 1$. Now, inscribe ΔXYM in ΔABC so that M is the midpoint of $\overline{BC}$, and X and Y lie between A and B and between A and C, respectively. Find the minimum perimeter possible for the inscribed triangle, ΔXYM.

SOLUTIONS

POLYNOMIALS IN THE Q MATRIX

B-244 Proposed by J. L. Hunsucker, University of Georgia, Athens, Georgia.

Let Q be the 2×2 matrix

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

and let

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

be the sum of a finite number of matrices chosen from the sequence Q, Q^2, Q^3, $\cdots$. Prove that $b = c$ and $a = b + d$.

Solution by Graham Lord, Temple University, Philadelphia, Pennsylvania.

It will be sufficient to show for $n = 1, 2, 3, \cdots$ that if

$$Q^n = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

then $b = c$ and $a = b + d$. For if each Q^n has this property then the sum of a finite number of terms from the sequence Q, Q^2, Q^3, $\cdots$ will retain the same property.

However, if

$$Q = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

then it is easily shown by induction that

$$Q^n = \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix}$$

for $n \geq 1$, and clearly this latter matrix has the required property.

Also solved by Richard Blazej, Wray G. Brady, Paul S. Bruckman, Warren Cheves, C. B. A. Peck, Richard W. Sielaff, Tony Waters, Gregory Wulczyn, David Zeitlin, and the Proposer.

SUMS AND DIFFERENCES OF FIBONACCI SQUARES

B-245 Proposed by Richard M. Grassl, University of New Mexico, Albuquerque, New Mexico.

Show that each term F_n with $n > 0$ in the sequence $F_0, F_1, F_2, \cdots$ is expressible as $x^2 + y^2$ or $x^2 - y^2$ with x and y terms of the sequence with distinct subscripts.

Solution by David Zeitlin, Minneapolis, Minnesota.

The result follows by noting that $F_{2n} = F_{n+1}^2 - F_{n-1}^2$ and $F_{2n-1} = F_n^2 + F_{n-1}^2$.

Also solved by Richard Blazej, W. G. Brady, Paul S. Bruckman, Warren Cheves, Herta T. Freitag, Graham Lord, C. B. A.Peck, Gregory Wulczyn, and the Proposer.

AT MOST ONE IS RATIONAL

B-246 Proposed by L. Carlitz, Duke University, Durham, North Carolina.

Show that at least one of the following sums is irrational.

$$\sum_{n=0}^{\infty} \frac{1}{F_{2n+1}}, \qquad \sum_{n=0}^{\infty} \frac{(-1)^n}{L_{2n+1}}.$$

Solution by C. B. A. Peck, State College, Pennsylvania.

Since (FQ, Vol. 5, pp. 469-471) sum I is $\sqrt{5}$ times sum II, sum I is irrational if sum II is rational, completing the proof.

Also solved by Paul S. Bruckman and the Proposer.

LUCAS MULTIPLES OF FIBONACCI NUMBERS

B-247 Proposed by Larry Lang, Student, San Jose State University, San Jose, California.

Given that m and n are integers with $0 < n < m$ and $F_n | L_m$, prove that n is 1, 2, 3, or 4.

Solution by Phil Mana, University of New Mexico, Albuquerque, New Mexico.

Let $m = qn + r$ with m, n, and q positive integers and $0 \le r < n$. Since

$$\gcd(F_n, F_{n+1}) = 1 \quad \text{and} \quad L_m = L_{m-n}F_{n+1} + L_{n-n-1}F_n ,$$

$F_n | L_m$ implies $F_n | L_{m-n}$. Continuing this way, one shows that $F_n | L_m$ implies $F_n | L_{m-qn}$, i.e., $F_n | L_r$. Then $F_n < L_r$, $r < n$, and $n > 4$ imply $r = n - 1$ since it is easily shown by induction that $F_n > L_r$ for $n > 4$ and $r < n - 1$. Since $L_{n-1} = F_n + F_{n-2}$, $F_n | L_{n-1}$ implies $F_n | F_{n-2}$. This is impossible for $n > 2$, completing the proof.

SOME CASES OF $n|F_n$

B-248 Proposed by V. E. Hoggatt, Jr., San Jose State University, San Jose, California.

Let k be a positive integer and let $h = 5^k$. Prove that $h|F_h$.

Solution by Graham Lord, Temple University, Philadelphia, Pennsylvania.

Proof by induction;

Let $h|F_h$ for $k = n$, and note that for $n = 1$, $h = 5|F_5 = 5$. The factorization $x^5 - y^5 = (x - y)(x^4 + x^3y + x^2y^2 + xy^3 + y^4)$ with $x = \alpha^h$ and $y = \beta^h$ yields

$$F_{5h} = F_h(L_{4h} - L_{2h} + 1) .$$

But $L_{4h} - L_{2h} + 1 = (5F_{2h}^2 + 2) - (5F_h^2 - 2) + 1 \equiv 0 \pmod 5$. ($I_{16}$, I_{17}, p. 59 of Hoggatt's book). Hence F_{5h} is divisible by $5h$ if F_h is divisible by h, which completes the induction.

Also solved by W. G. Brady, Paul S. Bruckman, Warren Cheves, Herta T. Freitag, Gregory Wulczyn, and the Proposer.

EXAMPLES OF $n|L_n$

B-249 Proposed by V. E. Hoggatt, Jr., San Jose State University, San Jose, California.

Let k be a positive integer and let $g = 2\cdot 3^k$. Prove that $g|L_g$.

Solution by Graham Lord, Temple University, Philadelphia, Pennsylvania.

It will be shown that if k is a positive integer and $g = 2\cdot 3^k$ then $(3g)|L_g$ but $(9g)\nmid L_g$, which implies the property asked in B-249.

Proof by induction.

Let the induction hypothesis be for $k = n$, $(3g)|L_g$ but $(9g)\nmid L_g$. For $n = 1$ the hypothesis is true since $3g = 18 = L_6$. From the induction hypothesis $L_g = 3gt$, where 3 and t are coprime. Then

$$\begin{aligned} L_{3g} &= L_g(L_{2g} - 1) && [\text{from } x^3 + y^3 = (x + y)(x^2 + xy + y^2)] \\ &= 3gt(L_g^2 - 3) && (I_{15}, \text{ p. 59, of Hoggatt's book}) \\ &= 9gt(3g^2t^2 - 1) , \end{aligned}$$

which shows that $[3(3g)]|L_{3g}$ but $[9(3g)]\nmid L_{3g}$.

Also solved by Paul S. Bruckman, Warren Cheves, Herta T. Freitag, Gregory Wulczyn, David Zeitlin, and the Proposer.