A POLYNOMIAL WITH GENERALIZED FIBONACCI COEFFICIENTS

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In Elementary Problem B-135 (this Quarterly, Vol. 6, No. 1, p. 90), L. Carlitz asks readers to show that

$$\sum_{k=0}^{n-1} F_k 2^{n-k-1} = 2^n - F_{n+2} , \tag{1}$$

and that

$$\sum_{k=0}^{n-1} L_k 2^{n-k-1} = 3(2^n) - L_{n+2} . \tag{2}$$

The problem invites generalization in at least two ways. It is natural to investigate

$$\sum_{k=0}^{n-1} T_k 2^{n-k-1} ,$$

where T_k is the generalized Fibonacci sequence with $T_1 = a$ and $T_2 = b$. It is not difficult to show by induction that

$$\sum_{k=0}^{n-1} T_k 2^{n-k-1} = T_2(2^n) - T_{n+2} . \tag{3}$$

The relations given in (1) and (2) are, thus, a consequence of (3).

A second generalization may be obtained by trying to determine whether anything worthwhile can be said about the polynomial

$$\sum_{k=0}^{n-1} T_k x^{n-k-1} . \tag{4}$$

This seems to be a more difficult problem than that posed by the first generalization, and the rest of this note is devoted to it.

To begin with, evaluating (4) for several values of n suggests that

$$\sum_{k=0}^{n-1} T_k x^{n-k-1} = a \sum_{k=0}^{n-1} F_{k-2}\, x^{n-k-1} + b \sum_{k=0}^{n-1} F_{k-1}\, x^{n-k-1} \quad . \tag{5}$$

This can be proved by induction. For $n = 1$, both members of (5) reduce to $b - a = T_0$. (We use $x^0 \equiv 1$ here.) If we now suppose (5) true for some integer $n \geq 1$, then

$$\sum_{k=0}^{n} T_k\, x^{n-k} = x \sum_{k=0}^{n-1} T_k\, x^{n-k-1} + T_n$$

$$= x \left[a \sum_{k=0}^{n-1} F_{k-2}\, x^{n-k-1} + b \sum_{k=0}^{n-1} F_{k-1}\, x^{n-k-1} \right] + T_n$$

and, since $T_n = a F_{n-2} + b F_{n-1}$,

$$\sum_{k=0}^{n} T_k\, x^{n-k} = a \sum_{k=0}^{n-1} F_{k-2}\, x^{n-k} + a F_{n-2}$$

$$+ b \sum_{k=0}^{n-1} F_{k-1}\, x^{n-k} + b F_{n-1}$$

$$= a \left[\sum_{k=0}^{n-1} F_{k-2}\, x^{n-k} + F_{n-2} \right] + b \left[\sum_{k=0}^{n-1} F_{k-1}\, x^{n-k} + F_{n-1} \right]$$

$$= a \sum_{k=0}^{n} F_{k-2}\, x^{n-k} + b \sum_{k=0}^{n} F_{k-1}\, x^{n-k} \quad .$$

This completes the proof of (5). The problem has, thus, been reduced slightly to the problem of evaluating an expression such as

$$\sum_{k=1}^{n} F_k\, x^{n-k}$$

in closed form, for such a result would lend some significance to the right member of (5).

Let us define

$$f_n(x) = \sum_{k=1}^{n} F_k x^{n-k} = x^n \sum_{k=1}^{n} \frac{F_k}{x^k} .$$

Now, it is known [1, p. 43] that the power series

$$\sum_{k=1}^{\infty} F_k t^{k-1}$$

converges to

$$\frac{1}{1 - t - t^2} .$$

The radius of convergence is

$$\lim_{k \to \infty} \frac{F_k}{F_{k+1}} = \frac{1}{\phi} ,$$

where

$$\phi = \frac{1 + \sqrt{5}}{2}$$

is the Golden Ratio. Thus, for a fixed value of t in the interval of convergence

$$-\frac{\sqrt{5} - 1}{2} < t < \frac{\sqrt{5} - 1}{2} ,$$

it follows that

$$\frac{1}{1 - t - t^2} = \sum_{k=1}^{\infty} F_k t^{k-1} = \sum_{k=1}^{n} F_k t^{k-1} + R_n ,$$

where $R_n \to 0$ as $n \to \infty$. Thus,

$$\sum_{k=1}^{n} F_k t^{k-1} = \frac{1}{1 - t - t^2} - R_n$$

or, what is the same,

$$\sum_{k=1}^{n} F_k t^k = \frac{t}{1 - t - t^2} - tR_n .$$

If we now let $t = 1/x$ then, for $x < -\phi$ or $x > \phi$,

$$\sum_{k=1}^{n} \frac{F_k}{x^k} = \frac{\frac{1}{x}}{1 - \frac{1}{x} - \frac{1}{x^2}} - \frac{1}{x} R_n = \frac{x}{x^2 - x - 1} - \frac{1}{x} R_n ,$$

and

$$x^n \sum_{k=1}^{n} \frac{F_k}{x^k} = \frac{x^{n+1}}{x^2 - x - 1} - x^{n-1} R_n .$$

We have, therefore,

$$f_n(x) = \frac{x^{n+1}}{x^2 - x - 1} - x^{n-1} R_n . \tag{6}$$

The problem is thus essentially reduced to finding the remainder R_n in some suitable form. Investigating (6) for the first few values of n suggests that

$$R_n = \frac{F_{n+1} x + F_n}{x^{n-1}(x^2 - x - 1)} .$$

This, in turn, suggests that

$$f_n(x) = \frac{x^{n+1}}{x^2 - x - 1} - x^{n-1}\left[\frac{F_{n+1} x + F_n}{x^{n-1}(x^2 - x - 1)}\right] .$$

That is,

$$\sum_{k=1}^{n} F_k x^{n-k} = \frac{x^{n+1} - F_{n+1} x - F_n}{x^2 - x - 1} \quad \text{for} \quad x \neq \frac{1 \pm \sqrt{5}}{2} . \tag{7}$$

We will prove (7) by induction. For $n = 1$, both members reduce to 1. If (7) is true for some integer $n \geq 1$, then

$$\sum_{k=1}^{n+1} F_k x^{n-k+1} = \sum_{k=1}^{n} F_k x^{n-k+1} + F_{n+1}$$

$$= x \sum_{k=1}^{n} F_k x^{n-k} + F_{n+1}$$

$$= x \frac{x^{n+1} - F_{n+1}x - F_n}{x^2 - x - 1} + F_{n+1}$$

$$= \frac{x^{n+2} - F_{n+1}x^2 - F_n x + F_{n+1}x^2 - F_{n+1}x - F_{n+1}}{x^2 - x - 1}$$

$$= \frac{x^{n+2} - (F_n + F_{n+1})x - F_{n+1}}{x^2 - x - 1}$$

$$= \frac{x^{n+2} - F_{n+2}x - F_{n+1}}{x^2 - x - 1} .$$

This completes the proof of (7).

Now, returning to the summations in (5),

$$\sum_{k=0}^{n-1} F_{k-2}x^{n-k-1} = F_{-2}x^{n-1} + F_{-1}x^{n-2} + F_0 x^{n-3} + \sum_{k=3}^{n-1} F_{k-2}x^{n-k-1}$$

$$= -x^{n-1} + x^{n-2} + \frac{1}{x^3}\sum_{k=3}^{n-1} F_{k-2}\,x^{n-k+2}$$

$$= -x^{n-1} + x^{n-2}$$

$$+ \frac{1}{x^3}\left[\sum_{k=3}^{n+2} F_{k-2}x^{n-k+2} - F_{n-2}x^2 - F_{n-1}x - F_n\right].$$

Using the change of variable $j = k - 2$ in the summation on the right, we have

$$\sum_{k=0}^{n-1} F_{k-2}x^{n-k-1} = -x^{n-1} + x^{n-2} + \frac{1}{x^3}\sum_{j=1}^{n} F_j x^{n-j}$$

$$- \frac{F_{n-2}x^2 + F_{n-1}x + F_n}{x^3} .$$

After substituting from (7), combining fractions and simplifying, the result is that

$$\sum_{k=0}^{n-1} F_{k-2}\, x^{n-k-1} = \frac{x^n(2 - x) - F_{n-2}x - F_{n-3}}{x^2 - x - 1}\,. \tag{8}$$

In a similar manner, we can use (7) to show that

$$\sum_{k=0}^{n-1} F_{k-1}\, x^{n-k-1} = \frac{x^n(x - 1) - F_{n-1}x - F_{n-2}}{x^2 - x - 1}\,. \tag{9}$$

Now substitute (8) and (9) into (5), combine fractions and arrange the numerator in powers of x. The result is

$$\sum_{k=0}^{n-1} T_k\, x^{n-k-1} = \frac{1}{x^2 - x - 1}\left\{x^n\left[(b - a)x + (2a - b)\right] - \left[aF_{n-2} + bF_{n-1}\right]x - \left[aF_{n-3} + bF_{n-2}\right]\right\}.$$

Consequently, we have the following generalization from Carlitz' problem:

$$\sum_{k=0}^{n-1} T_k x^{n-k-1} = \frac{(T_0 + T_{-1})x^n - T_n x - T_{n-1}}{x^2 - x - 1}\,. \tag{10}$$

It is not difficult to see that (10) reduces to (3) when $x = 2$. Other results of interest can be obtained by letting $x = \pm 1$ in (10).

REFERENCE

1. Brother Alfred Brousseau, An Introduction to Fibonacci Discovery, The Fibonacci Association, 1965.