**The Z Transform and the Fibonacci Sequence**

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**Definition.** The Z transform of \( f(n) \) is the function

\[
\zeta[f(n)] = F(z) = \sum_{n=0}^{\infty} f(n)z^{-n}, \quad |z| > \frac{1}{\rho}
\]

where \( z \) is a complex variable and \( \rho \) is the radius of convergence of the series.

Applying the Z transform to the recursion relation

\[
f_{n+2} = f_{n+1} + f_n,
\]

we obtain

\[
\zeta[f_{n+2}] = \zeta[f_{n+1} + f_n] = \zeta[f_{n+1}] + \zeta[f_n].
\]

Using the shifting theorem for Z transforms,

\[
\zeta[f(n + m)] = z^m [F(z) - F_m(z)],
\]

where

\[
F_m(z) = \sum_{k=0}^{m-1} f(k)z^{-k},
\]

which yields

\[
z^2[F(z) - F_1(z)] = z[F(z) - F_1(z)] + F(z)
\]

and

\[
(z^2 - z - 1)F(z) = z^2F_1(z) - zF_1(z).
\]

Hence
\[ F(z) = \frac{z^2[f(0) - f(1)z^{-1}] - z[f(0)]}{z^2 - z - 1}, \]

where

\[ z^2 - z - 1 \neq 0. \]

Since \( f_0 = 0 \) and \( f_1 = 1 \), we have

\[ F(z) = \frac{z}{z^2 - z - 1}. \]

\( F(z) \) is a Laurent series. Therefore, we can multiply \( F(z) \) by \( z^{n-1} \) and integrate it around a circle for which \( |z| > R \). This gives

\[ \int \frac{F(z)z^{n-1}dz}{\Gamma} = 2\pi i f(n) \]

or

\[ f(n) = \frac{1}{2\pi i} \int \frac{F(z)z^{n-1}dz}{\Gamma} = \sum \text{Residues of } F(z)z^{n-1}. \]

Hence

\[ f(n) = \sum \text{Residues} \left[ \frac{z}{(z - \frac{1 + \sqrt{5}}{2})(z - \frac{1 - \sqrt{5}}{2})} \right] z^{n-1} \]

\[ = \lim_{z \to \frac{1 + \sqrt{5}}{2}} \left[ \frac{z^n}{z - \frac{1 - \sqrt{5}}{2}} \right] + \lim_{z \to \frac{1 - \sqrt{5}}{2}} \left[ \frac{z^n}{z - \frac{1 + \sqrt{5}}{2}} \right] \]

\[ = \left( \frac{1 + \sqrt{5}}{2} \right)^n / \sqrt{5} - \left( \frac{1 - \sqrt{5}}{2} \right)^n / \sqrt{5}. \]

Therefore

\[ f(n) = (\alpha^n - \beta^n) / \sqrt{5}, \]

where

\[ \alpha^n = \left( \frac{1 + \sqrt{5}}{2} \right)^n \]

and

\[ \beta^n = \left( \frac{1 - \sqrt{5}}{2} \right)^n, \]

which is Binet's formula.