MORE HIDDEN HEXAGON SQUARES

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In [1], Hoggatt and Hansell prove the following remarkable result.

<u>Theorem 1.</u> Let $\binom{m}{n}$ be such that $0 < n < m$ and $2 \le m$. Then the product of the six binomial coefficients surrounding $\binom{m}{n}$ is a perfect integral square.

In this paper, we show that this theorem is a special case of a more general result. In particular, we prove the following theorem.

<u>Theorem 2.</u> Let H_j, for j odd, be a hexagon of entries from Pascal's triangle with $j + 1$ entries per side and with the sides lying along main diagonal and horizontal rows of the triangle. Then the product of the entries forming H_j is an integral square.

<u>Proof.</u> Let j be a positive odd integer and let n and r be integers with $1 \le n - j$, $j \le r \le n$, and $0 \le r \le n - j$. If H_j is centered at $\binom{n}{r}$, then it can be displayed in the following way where we label the sides I, $\cdots$, VI.

$$
\begin{array}{ccccc}
 & & \binom{n-j}{r-j}\binom{n-j}{r-j+1}\cdots\binom{n-j}{r-1}\binom{n-j}{r} & & \\
 & \binom{n-j+1}{r-j} & \text{I} & \binom{n-j+1}{r+1} & \\
 & \cdot^{\cdot^{\cdot}} \quad \text{VI} & & \text{II} \quad \ddots & \\
\binom{n-1}{r-j} & & & & \binom{n-1}{r+j-1} \\
\binom{n}{r-j} & & & & \binom{n}{r+j} \\
\binom{n+1}{r-j+1} & & & & \binom{n+1}{r+j} \\
 & \ddots \quad \text{V} & & \text{III} \quad \cdot^{\cdot^{\cdot}} & \\
 & \binom{n+j-1}{r-1} & & \binom{n+j-1}{r+j} & \\
 & & \text{IV} & & \\
 & & \binom{n+j}{n}\binom{n+j}{r+1}\cdots\binom{n+j}{r+j-1}\binom{n+j}{r+j} & &
\end{array}
$$

Of course, each coefficient is of the form $\frac{a}{bc}$ where a, b, and c are the appropriate factorials. We prove that the desired product is a square by proving that the product of the a's is a square and similarly for the b's and c's. The products of the a's in sides I and IV, respectively, are clearly $[(n - j)!]^{j+1}$ and $[(n + j)!]^{j+1}$ and both are squares since j is odd. Also, the product of the a's in II, III, V, and VI and not in I or IV is clearly

$$[(n-j+1)!(n-j+2)!\cdots(n+j-1)!]^2 .$$

Similarly, the products of the b's in III and VI, respectively, are $[(r+j)!]^{j+1}$ and $[(r-j)!]^{j+1}$, and the product of the b's in I, II, IV and V and not in III and VI is

$$[(r-j+1)!(r-j+2)!\cdots(r+j-1)!]^2 .$$

Finally, the products of the c's in II and V, respectively, are $[(n-r-j)!]^{j+1}$ and $[(n-r+j)!]^{j+1}$ and the product of the c's in I, III, IV and VI and not in II and V is

$$[(n-j-r+1)!(n-j-r+2)!\cdots(n+j-r-1)!]^2 .$$

Therefore, the product of the coefficients in question is a rational square and, since the product is a product of integers, it is also an integral square as claimed.

REFERENCE

1. V. E. Hoggatt, Jr., and Walter Hansell, "The Hidden Hexagon Squares," Fibonacci Quarterly, Vol. 9 (1971), pp. 120, 133.

THE BALMER SERIES AND THE FIBONACCI NUMBERS

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In 1885, J. J. Balmer discovered that the wave lengths (λ) of four lines in the hydrogen spectrum (now known as "Balmer Series") can be expressed by the multiplication of a numerical constant $k = 364.5$ nm (1 nm = 1 nanometre = 10^{-9} m) by the simple fractions as follows:

$$656 = \frac{9}{5} \times 364.5 \tag{1}$$

$$486 = \frac{4}{3} \times 364.5 = \frac{16}{12} \times 364.5 \tag{2}$$

$$434 = \frac{25}{21} \times 364.5 \tag{3}$$

$$410 = \frac{9}{8} \times 364.5 = \frac{36}{32} \times 364.5 . \tag{4}$$

By extending both fractions, 4/3 and 9/8, he recognized the successive numerators as the squares 3^2, 4^2, 5^2 and 6^2, and the denominators as the square-differences $3^2 - 2^2$, $4^2 - 2^2$, $5^2 - 2^2$, $6^2 - 2^2$.

From this he developed his famous formula:

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