A METHOD FOR CONSTRUCTING SINGLY EVEN MAGIC SQUARES

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In a recent note* we described a method for constructing magic squares of order $n = 2(2m + 1)$ based on systematic alteration of 2×2 blocks of integers substituted for the integers of any odd square of order $2m + 1$. The present note derives a convenient alternative rule starting from a block of four odd squares of order $2m + 1$. Its derivation shows the existence of a very large number of similar rules.

Divide the square of order $n = 2\ (2m + 1)$, with sum

$$S_n = n(n^2 + 1)/2 = 2S_{2m+1} + 3(2m + 1)^3$$

into four squares of order $2m + 1$. Label them I, II, III, IV as shown in Fig. 1, filling the cells of I with integers of any magic square of order $2m + 1$, filling II with any square of the same order whose integers have each been augmented by $(2m + 1)^2$, likewise III and IV,

I	III
IV	II

Figure 1

where the augmentations are respectively by $2(2m + 1)^2$ and $3(2m + 1)^2$, and the unaugmented squares of IV and I are identical, likewise those of II and III. Clearly the column sums each add up to S_n, and this property is not destroyed by interchanges within a column.

The upper $(2m + 1)$ rows sum to $2S_{2m+1} + 2(2m + 1)^3$, while the lower $(2m + 1)$ rows sum to $2S_{2n+1} + 4(2m + 1)^3$. Exchanges within columns which reduce the lower rows by $(2m + 1)^3$ and increase the upper rows by the same amount will thus bring the row sum to S_n. If p interchanges are made between I and IV and q between II and III, all of them in the same row, then the upper row increases by $(3p - q)(2m + 1)^2$, the lower row decreasing by the same amount. Any p and q less than $2m + 1$ satisfying $3p - q = 2m + 1$ will bring the row sum to S_n. For k an integer, positive, zero or negative, and satisfying $-2m + 1 \leq 3k \leq m + 2$ we have $p = m + k$, $q = m + 3k - 1$ as the possible cases. The case $p = m$, $q = m - 1$ is the simplest.

The two diagonal sums differ by $4(2m + 1)^3$, or twice the row difference. As the row sum adjustments are independent of which cells in a row are selected for the $p + q$ inter-

*J. Rothstein, American Math. Monthly, Vol. 67, No. 6, pp. 583-585 (June-July, 1960).

changes, we select them to bring the diagonal sums to S_n. If m diagonal cells of I interchange with the corresponding (non-diagonal) cells of IV, likewise m diagonal cells of IV with the corresponding (non-diagonal) cells of I, and the center cells of I and IV are also interchanged, then the I-II diagonal increases by $(2m + 1)2(2m + 1)^2$ and the III-IV diagonal decreases by the same amount, thus bringing them to S_n. This diagonal correction, which uses only I-IV interchanges, applies only if $p \geq m$. Other rules, involving II-III interchanges also, can easily be worked out.

Figure 2 gives a pictorial representation of a simple rule for $p = m$, with I-IV diagonal correction, illustrated for the case $m = 2$. The numbers assigned to the empty cells of the squares of order $2m + 1$ are left undisturbed. Those assigned to cells with + or - are interchanged with the numbers in the corresponding cells, i.e., the number in cell (i, j) of I exchanges with that in cell (i, j) of IV, likewise II and III. A - label can be moved anywhere in its row (in its square of order $2m + 1$) except to a cell on a diagonal. A + label, except for those in the center cells, which are fixed, can be displaced to the other diagonal position in its row as long as the same number of mobile + labels are on the diagonal of the square of order n as off it (these are still on the diagonals of I and IV, of course). It is understood that when a label moves, the corresponding label moves correspondingly. In Fig. 2, it can be seen that a simple rule can be expressed as follows. After I, II, III, IV have been written down, interchange the center elements and the m columns on the left, with the exception of the center cell of one column, between I and IV. Perform the same interchanges between II and III except that diagonal cells are not interchanged.

+	-					-			
-	+				-				
	-	+				-			
-	+				-				
+	-					-			
+	-					-			
-	+				-				
	-	+				-			
-	+				-				
+	-					-			

Figure 2