# ON GENERALIZED FIBONACCI QUARTERNIONS

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Horadam [1] defined and studied in detail the generalized Fibonacci sequence defined by

$$H_n = H_{n-1} + H_{n-2} \quad (n \geq 3), \quad \text{with} \quad H_1 = p, \quad H_2 = p + q, \tag{1}$$

$p$ and $q$ being arbitrary integers. In a later article [2], he defined Fibonacci and generalized Fibonacci quaternions as follows, and established a few relations for these quaternions:

$$P_n = H_n + iH_{n+1} + jH_{n+2} + kH_{n+3} \tag{2}$$

$$Q_n = F_n + iF_{n+1} + jF_{n+2} + kF_{n+3}, \tag{3}$$

where

$$i^2 = j^2 = k^2 = -1, \quad ij = -ji = k, \quad jk = -kj = i, \quad ki = -ik = j, \tag{4}$$

and $F_n$ is the $n^{th}$ Fibonacci number. He also defined the conjugate quaternion as

$$\overline{P}_n = H_n - iH_{n+1} - jH_{n+2} - kH_{n+3} \tag{5}$$

and $\overline{Q}_n$ in a similar way.

We shall now establish a few interesting relations for these quaternions. Let $R_n$ be the quaternion for the generalized sequence $M_n$ defined by

$$M_n = M_{n-1} + M_{n-2} \quad (n \geq 3), \quad \text{with} \quad M_1 = r, \quad M_2 = r + s. \tag{6}$$

Then from (2) and (5),

$$\overline{P}_n = 2H_n - P_n. \tag{7}$$

Also,

$$\overline{R}_n = 2M_n - R_n. \tag{8}$$

Hence

$$P_n\overline{R}_n - \overline{P}_nR_n = 2(M_nP_n - H_nR_n)\ . \tag{9}$$

Similarly, the following results may be obtained:

$$P_n\overline{R}_n + \overline{P}_nR_n = 2(2M_nP_n + 2H_nR_n - P_nR_n)$$

$$P_nR_n - \overline{P}_n\overline{R}_n = 2(H_nR_n - 2H_nM_n + M_nP_n)$$

$$P_n\overline{R}_n + P_n\overline{R}_n = \overline{R}_nP_n + \overline{P}_nR_n$$

$$P_n\overline{R}_n - \overline{P}_nR_n = \overline{R}_nP_n - R_n\overline{P}_n = 2(M_nP_n - H_nR_n)$$

$$P_n\overline{R}_n - \overline{R}_nP_n = \overline{P}_nR_n - R_n\overline{P}_n = R_nP_n - P_nR_n\ .$$

It may also be seen that $P_nR_n \neq R_nP_n$ unless $P_n = R_n$, whereas,

$$P_n\overline{P}_n = \overline{P}_nP_n = 2H_nP_n - P_n^2\ . \tag{10}$$

Some of these results have been obtained earlier [3] for $P_n$ and $Q_n$, which may be deduced by assuming $r = 1$, $s = 0$ in which case $M_n = F_n$ and $R_n = Q_n$. Now consider

$$\begin{aligned} &F_{m+1}P_{n+1} + F_mP_n \\ &\qquad = (F_{m+1}H_{n+1} + F_mH_n) + i(F_{m+1}H_{n+2} + F_mH_{n+1}) \\ &\qquad\quad + j(F_{m+1}H_{n+3} + F_mH_{n+2}) + k(F_{m+1}H_{n+4} + F_mH_{n+3}). \end{aligned}$$

It is also known [1] that

$$H_{m+n+1} = F_{m+1}H_{n+1} + F_mH_n = F_{n+1}H_{m+1} + F_nH_m\ . \tag{11}$$

Hence we have

$$\begin{aligned} F_{m+1}P_{n+1} + F_mP_n &= H_{m+n+1} + iH_{m+n+2} + jH_{m+n+3} + kH_{m+n+4} \\ &= P_{m+n+1}\ . \end{aligned}$$

Thus,

$$P_{m+n+1} = F_{m+1}P_{n+1} + F_mP_n = F_{n+1}P_{m+1} + F_nP_m\ . \tag{12}$$

Also

$$P_{2n+1} = F_{n+1}P_{n+1} + F_nP_n \tag{13}$$

and

$$P_{2n} = F_{n+1}P_n + F_nP_{n-1} = F_nP_{n+1} + F_{n-1}P_n \quad . \tag{14}$$

It may also be verified that

$$P_n\overline{P}_n = \overline{P}_nP_n = 3(2p - q)H_{2n+3} - (p^2 - pq - q^2)F_{2n+3} \quad , \tag{15}$$

where use has been made of the relation [1]

$$H_{n+1} = qF_n + pF_{n+1} \quad . \tag{16}$$

Hence from (15) and (16),

$$\begin{aligned} P_n\overline{P}_n = \overline{P}_nP_n &= 3(2pq - q^2)F_{2n+2} + (p^2 + q^2)F_{2n+3} \\ &= 3(p^2F_{2n+3} + 2pqF_{2n+2} + q^2F_{2n+1}) \quad . \end{aligned} \tag{17}$$

Hence

$$P_n\overline{P}_n + P_{n-1}\overline{P}_{n-1} = 3(p^2L_{2n+2} + 2pqL_{2n+1} + q^2L_{2n}) \quad . \tag{18}$$

Also from (12) we have

$$P_n^2 + P_{n-1}^2 = 2(H_nP_n + H_{n-1}P_{n-1}) - (P_n\overline{P}_n + P_{n-1}\overline{P}_{n-1}) \quad .$$

Using (13) and (21) we get

$$P_n^2 + P_{n-1}^2 = 2P_{2n-1} - 3(p^2L_{2n+2} + 2pqL_{2n+1} + q^2L_{2n}) \quad . \tag{19}$$

If $p = 1$, $q = 0$ then we have the Fibonacci sequence $F_n$ and the corresponding quarternion $Q_n$ for which we may write the following results:

$$Q_n\overline{Q}_n = \overline{Q}_nQ_n = 3F_{2n+3} \tag{20}$$

$$Q_n\overline{Q}_n + Q_{n-1}\overline{Q}_{n-1} = 3L_{2n+2} \tag{21}$$

$$Q_n^2 + Q_{n-1}^2 = 2Q_{2n-1} - 3L_{2n+2} \quad . \tag{22}$$

Similar results may be obtained for the Lucas numbers and its quarternion by letting $p = 1$ and $q = 2$ in the various results derived in this article. Also, many other interesting results for these quarternions $P_n$ and $M_n$ may be obtained.

## REFERENCES

1. A. F. Horadam, "A Generalized Fibonacci Sequence," Amer. Math. Monthly, 68 (1961), pp. 455-459.
2. A. F. Horadam, "Complex Fibonacci Numbers and Fibonacci Quarternions," Amer. Math. Monthly, 70 (1963), pp. 289-291.
3. M. R. Iyer, "A Note on Fibonacci Quarternions," to be published.