GEOMETRIC PROOF OF A RESULT OF LEHMER'S

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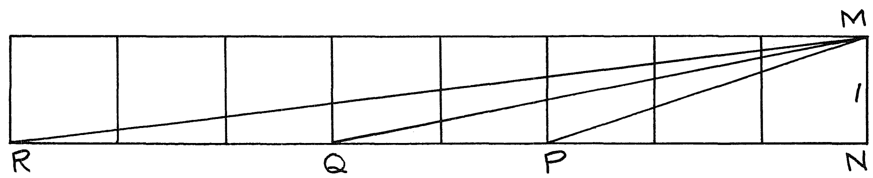
The rectangle in the figure is composed of unit squares: $NP = F_{2n}$, $NQ = F_{2n+1}$, $NR = F_{2n+2}$ and $MN = 1$. It follows that $MP = (F_{2n}^2 + 1)^{1/2}$, $PQ = F_{2n+1} - F_{2n}$, and $PR = F_{2n+2} - F_{2n}$.



Starting with the well-known identity,

$$F_{2n+1}F_{2n+2} - F_{2n}F_{2n+3} = 1,$$

we have

$$F_{2n+1}F_{2n+2} - F_{2n}(F_{2n+1} + F_{2n+2}) + F_{2n}^2 = F_{2n}^2 + 1$$

$$(F_{2n+1} - F_{2n})(F_{2n+2} - F_{2n}) = F_{2n}^2 + 1$$

$$(F_{2n+1} - F_{2n}) : (F_{2n}^2 + 1)^{1/2} = (F_{2n}^2 + 1)^{1/2} : (F_{2n+2} - F_{2n}) .$$

Therefore, triangles QPM and MPR are similar, since the sides including their common angle are proportional. Therefore $\angle MRP = \angle QMP$. It follows that $\angle MPN = \angle QMP + \angle MQP = \angle MRP + \angle MQP$. That is, $\operatorname{arccot} F_{2n} = \operatorname{arccot} F_{2n+1} + \operatorname{arccot} F_{2n+2}$. Thus we write:

$$\begin{aligned}
\operatorname{arccot} 1 &= \operatorname{arccot} 2 + \operatorname{arccot} 3 \\
&= \operatorname{arccot} 2 + \operatorname{arccot} 5 + \operatorname{arccot} 8 \\
&= \operatorname{arccot} 2 + \operatorname{arccot} 5 + \operatorname{arccot} 13 + \operatorname{arccot} 21 \\
&= \cdots \\
&= \sum_{i=1}^{n} \operatorname{arccot} F_{2i+1} + \operatorname{arccot} F_{2n+2} \\
&= \sum_{i=1}^{\infty} \operatorname{arccot} F_{2i+1} .
\end{aligned}$$

This result was announced by D. H. Lehmer [1] in 1936, and proved in different ways by M. A. Heaslet [2] and V. E. Hoggatt, Jr. [3,4]. The first value of arccot 1 above applies to Gardner's three-square problem [5] which has been proven synthetically in 54 ways [6]. Proof of the second value of arccot 1 is asked for in [7].

REFERENCES

1. D. H. Lehmer, Problem Proposal 3801, American Math. Monthly, 43 (Nov., 1936), p. 580.
2. M. A. Heaslet, Problem Solution 3801, American Math. Monthly, 45 (Nov., 1938), pp. 636-637.
3. Verner E. Hoggatt, Jr., and I. D. Ruggles, "A Primer for the Fibonacci Numbers, Part V," The Fibonacci Quarterly, 2 (Feb., 1964), pp. 59-65.
4. Verner E. Hoggatt, Jr., "Fibonacci Trigonometry," The Mathematical Log, Vol. XIII, No. 2, Dec., 1968, p. 3.
5. Martin Gardner, "Mathematical Games," Scientific American, 222, No. 2 (Feb., 1970), pp. 112-114; No. 3 (Mar., 1970), pp. 121-125.
6. Charles W. Trigg, "A Three-Square Geometry Problem," Journal of Recreational Math., April, 1971, pp. 90-99.
7. Alfred E. Neuman, Problem Proposal 243, Pi Mu Epsilon Journal, 6 (Fall, 1970), p. 133.

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$$\lambda = k \frac{n^2}{n^2 - 2^2} \qquad (n = 3, 4, 5, 6)$$

or in the better known form:

$$\nu = R\left(\frac{1}{2^2} - \frac{1}{n^2}\right) ,$$

where ν is the frequency and R the "Rydberg's constant."

It may be of interest to note that all denominators of the simple fractions used by Balmer for deriving his formula, i.e., 3, 5, 8 and 21, are Fibonacci numbers.