# IRREDUCIBILITY OF LUCAS AND GENERALIZED LUCAS POLYNOMIALS

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## 1. INTRODUCTION

In [5], Webb and Parberry discuss several divisibility properties for the sequence $\{F_n(x)\}$ of Fibonacci polynomials defined recursively by

$$(1) \qquad F_0(x) = 0, \quad F_1(x) = 1, \quad F_{n+2}(x) = xF_{n+1}(x) + F_n(x), \quad n \geq 0 .$$

In particular, Webb and Parberry prove that $F_p(x)$ is irreducible over the integral domain of the integers if and only if $p$ is a prime.

In [1], Bergum and Kranzler develop many relationships which exist between the sequence $\{F_n(x)\}$ of Fibonacci polynomials and the sequence $\{L_n(x)\}$ of Lucas polynomials defined recursively by

$$(2) \qquad L_0(x) = 2, \quad L_1(x) = x, \quad L_{n+2}(x) = xL_{n+1}(x) + L_n(x), \quad n \geq 0 .$$

Specifically, Bergum and Kranzler show that

$$(3) \qquad L_n(x) \mid L_m(x) \quad \text{iff} \quad m = (2k - 1)n, \quad k \geq 1.$$

With $n = 1$, we see that $x|L_n(x)$ for all odd integers $m$ so that the result of Webb and Parberry does not hold for the sequence $\{L_n(x)\}$.

In [4], Hoggatt and Long show that the result of Webb and Parberry does hold for the sequence $\{U_n(x,y)\}$ of generalized Fibonacci polynomials defined by the recursion

$$(4) \qquad U_0(x,y) = 0, \quad U_1(x,y) = 1, \quad U_{n+2}(x,y) = xU_{n+1}(x,y) + yU_n(x,y), \quad n \geq 0 .$$

The purpose of this paper is to obtain necessary and sufficient conditions for the irreducibility of the elements of the sequence $\{L_n(x)\}$ as well as the elements of the sequence $\{V_n(x,y)\}$ of generalized Lucas polynomials defined by the recursion

$$(5) \qquad V_0(x,y) = 2, \quad V_1(x,y) = x, \quad V_{n+2}(x,y) = xV_{n+1}(x,y) + yV_n(x,y), \quad n \geq 0 .$$

The first few terms of the sequence $\{V_n(x,y)\}$ are

| n | $V_n(x,y)$ |
|---|---|
| 1 | $x$ |
| 2 | $x^2 + 2y$ |
| 3 | $x^3 + 3xy$ |
| 4 | $x^4 + 4x^2y + 2y^2$ |
| 5 | $x^5 + 5x^3y + 5xy^2$ |
| 6 | $x^6 + 6x^4y + 9x^2y^2 + 2y^3$ |
| 7 | $x^7 + 7x^5y + 14x^3y^2 + 7xy^3$ |
| 8 | $x^8 + 8x^6y + 20x^4y^2 + 16x^2y^3 + 2y^4$ |
| 9 | $x^9 + 9x^7y + 27x^5y^2 + 30x^3y^3 + 9xy^4$ . |

Observe that $L_n(x) = V_n(x,1)$ so that with $y = 1$, we also have the first nine terms of the sequence $\{L_n(x)\}$.

## 2. IRREDUCIBILITY OF $L_n(x)$

The basic fact that we shall use is found in [2, p. 77] and is

Theorem 2.1. (Eisenstein's irreducibility criterion.) For a given prime $p$, let

$$F(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

be any polynomial with integral coefficients such that

$$a_{n-1} \equiv a_{n-2} \equiv \cdots \equiv a_0 \equiv 0 \pmod{p}, \quad a_n \not\equiv 0 \pmod{p}, \quad a_0 \not\equiv 0 \pmod{p^2}$$

then $F(x)$ is irreducible over the field of rationals.

To establish our first irreducibility theorem, we use the following.

Lemma 2.1. Every coefficient of $L_{2^n}(x)$, except for the leading coefficient, is divisible by 2 and 4 does not divide the constant term.

Proof. If $n = 1$ then $L_2(x) = x^2 + 2$ and the lemma is obviously true. Assume the lemma is true for $n$.

In [1], we find

$$L_{2k}(x) = L_k^2(x) - 2(-1)^k . \tag{6}$$

Hence,

$$L_{2^{n+1}}(x) = L_{2^n}^2(x) - 2 . \tag{7}$$

By the induction hypothesis, it is obvious that $L_{2^{n+1}}(x)$ is monic and every coefficient of $L_{2^{n+1}}(x)$ is divisible by 2. Furthermore, since $L_{2^n}(x)$ has constant term +2 we see that $L_{2^n}^2(x)$ has constant term +4, thus $L_{2^{n+1}}(x)$ has constant term +2. Therefore, the constant term of $L_{2^{n+1}}(x)$ is divisible by 2 but not by 4 and the lemma is proved.

An immediate result of Lemma 2.1 with the aid of Theorem 2.1 is

Theorem 2.2. The Lucas polynomial $L_{2^k}(x)$ is irreducible over the rationals for $k \geq 1$.

Although $L_p(x)$ is not irreducible if $p$ is a prime, we can show that $L_p(x)/x$ is irreducible for every odd prime $p$.

First we note, as is pointed out in [1], that

$$L_n(x) = \alpha^n + \beta^n \ , \tag{8}$$

where $\alpha = (x + \sqrt{x^2+4})/2$ and $\beta = (x - \sqrt{x^2+4})/2$. Hence, if $n = 2m+1$ we have

$$\begin{aligned} L_n(x) &= (x + \sqrt{x^2+4})^n/2^n + (x - \sqrt{x^2+4})^n/2^n \\ &= 2^{-n}\left(\sum_{k=0}^{n}\binom{n}{k}x^{n-k}(x^2+4)^{k/2} + \sum_{k=0}^{n}\binom{n}{k}(-1)^k x^{n-k}(x^2+4)^{k/2}\right) \\ &= 2^{-(n-1)}\sum_{k=0}^{m}\binom{n}{2k}x^{n-2k}(x^2+4)^k \\ &= 2^{-(n-1)}\sum_{k=0}^{m}\sum_{s=0}^{k}\binom{n}{2k}\binom{k}{s}x^{n-2s}2^{2s} \ . \end{aligned} \tag{9}$$

Therefore,

$$L_n(x)/x = 2^{-(n-1)}\sum_{k=0}^{m}\sum_{s=0}^{k}\binom{n}{2k}\binom{k}{s}x^{n-2s-1}2^{2s}, \qquad n = 2m+1 \ . \tag{10}$$

For each $s$, $0 \leq s \leq m$, we see that the coefficient of $x^{n-2s-1}$ is

$$2^{-(n-2s-1)}\sum_{k=s}^{m}\binom{n}{2k}\binom{k}{s}, \qquad n = 2m+1 \ . \tag{11}$$

When $s = 0$, we have the leading coefficient of $L_n(x)$ which is $1$ so that

$$2^{-(n-1)}\sum_{k=0}^{m}\binom{n}{2k}\binom{k}{0} = 1, \qquad n = 2m+1 \ . \tag{12}$$

When $s = m$ in (11), we have the constant term of $L_n(x)$ which is $n$. If we now let $n$ be an odd prime $p$ and recall that $p$ divides

$$\binom{p}{2k}$$

if $p$ is a prime, then $p$ is a factor of (11) for each value of $s$,

$$1 \le s \le \frac{(p-3)}{2} .$$

Hence, by Eisenstein's criterion, the following is true.

Theorem 2.3. The polynomials $L_p(x)/x$ are irreducible over the rationals if $p$ is an odd prime.

By (11) and the fact that the coefficients of $L_n(x)$ are integers, we have

Corollary 2.1. If $n = 2m + 1$ then $2^{n-2s-1}$ divides

$$\sum_{k=s}^{m} \binom{n}{2k}\binom{k}{s}$$

for any $s$ such that $0 \le s \le m$.

Using (3) together with Theorems 2.2 and 2.3, we have

Theorem 2.4. (a) The Lucas polynomials $L_n(x)$, $n \ge 1$, are irredicuble over the rationals if and only if $n = 2^k$ for some integer $k \ge 1$.

(b) The polynomials $L_n(x)/x$, $n$ odd, are irreducible over the rationals if and only if $n$ is a prime.

## 3. IRREDUCIBILITY OF $V_n(x,y)$

It is a well known fact that

$$U_n(x,y) = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \qquad n \ge 0 \tag{13}$$

and

$$V_n(x,y) = \alpha^n + \beta^n, \qquad n \ge 0 , \tag{14}$$

where $\alpha = (x + \sqrt{x^2 + 4y})/2$ and $\beta = (x - \sqrt{x^2 + 4y})/2$.

In [4], we find

Lemma 3.1. (a) For $n \ge 0$,

$$U_n(x,y) = \sum_{k=0}^{[(n-1)/2]} \binom{n-k-1}{k} x^{n-2k-1} y^k .$$

(b) For $n \ge 0$, $m \ge 0$,

$$(U_m(x,y),\ U_n(x,y)) = U_{(m,n)}(x,y) .$$

Using (13) and (14), a straightforward argument yields

Lemma 3.2. (a) $V_n(x,y) = yU_{n-1}(x,y) + U_{n+1}(x,y), \quad n \geq 1;$

(b) $U_{2n}(x,y) = U_n(x,y)V_n(x,y), \quad n \geq 0;$

(c) $U_{2n}(x,y)V_{(2k+1)n+1}(x,y) + y^{2n}V_{(2k-1)n}(x,y)$
$= V_{(2k+1)n}(x,y)U_{2n+1}(x,y).$

Using (a) of Lemma 3.1 and 3.2, we have, for $n \geq 1$, that

$$V_n(x,y) = \sum_{k=0}^{[(n-2)/2]} \binom{n-k-2}{k} x^{n-2k-2}y^{k+1} + \sum_{k=0}^{[n/2]} \binom{n-k}{k} x^{n-2k}y^k$$

$$(15) \qquad = \sum_{k=1}^{[n/2]} \binom{n-k-1}{k-1} x^{n-2k}y^k + \sum_{k=0}^{[n/2]} \binom{n-k}{k} x^{n-2k}y^k$$

$$= \sum_{k=1}^{[n/2]} \binom{n-k-1}{k-1} \frac{n}{k} x^{n-2k}y^k + x^n .$$

Hence,

Lemma 3.3. (a) For $n \geq 1$, $V_n(x,y^2)$ is homogeneous of degree $n$.

(b) If $n$ is odd then $x$ is a factor of $V_n(x,y^2)$ and $V_n(x,y^2)/x$ is homogeneous of degree $n - 1$.

By (b) of Lemma 3.1, $(U_{2n}(x,y), U_{2n+1}(x,y)) = 1$. Using this fact together with (b) of Lemma 3.2 and induction on $k$ in (c) of Lemma 3.2, one obtains

Lemma 3.4. If $k \geq 1$ then $V_n(x,y) | V_{(2k-1)n}(x,y)$.

In [3, p. 376, Problem 5], we find

Lemma 3.5. A homogeneous polynomial $f(x,y)$ over a field $F$ is irreducible over $F$ if and only if the corresponding polynomial $f(x,1)$ is irreducible over $F$.

Using Lemmas 3.3 and 3.5 with Theorem 2.4, we have

Theorem 3.1. (a) The polynomials $V_n(x,y^2)$ are irreducible over the rationals if and only if $n = 2^k$ for some integer $k \geq 1$.

(b) The polynomials $V_n(x,y^2)/x$, $n$ odd are irreducible over the rationals if and only if $n$ is an odd prime.

Since $f(x,y)$ is irreducible if $f(x,y^2)$ is irreducible and $x$ is a factor of $V_n(x,y)$ for $n$ odd by (15), we apply Lemma 3.4 and Theorem 3.1 to obtain

Theorem 3.2. (a) The polynomials $V_n(x,y)$ are irreducible over the rationals if and only if $n = 2^k$ for some integer $k$ greater than or equal to one.

(b) The polynomials $V_n(x, y)/x$, $n$ odd, are irreducible over the rationals if and only if $n$ is an odd prime.

Letting $y = 1$ and $n = 2m + 1$ in (15), we see that

$$L_n(x)/x = \sum_{k=1}^{m} \binom{n-k-1}{k-1} \frac{n}{k} x^{n-2k-1} + x^{n-1} . \tag{16}$$

Comparing the coefficients of $x^{n-2s-1}$ in (16), $1 \le s \le m$, with the result obtained in (11), we have

<u>Corollary 3.1</u>. If $n = 2m + 1$ and $1 \le s \le m$ then

$$2^{-(n-2s-1)} \sum_{k=s}^{m} \binom{n}{2k}\binom{k}{s} = \binom{n-s-1}{s-1} \frac{n}{s} .$$

## REFERENCES


1. G. Bergum and A. Kranzler, Linear Recurrences-Identities and Divisibility Properties, unpublished.
2. G. Birkhoff and S. MacLane, <u>A Survey of Modern Algebra</u>, The MacMillan Company, New York, N Y., 1965.
3. G. Birkoff and S. MacLane, <u>Algebra</u>, The MacMillan Company, New York, N. Y., 1967.
4. V. E. Hoggatt, Jr., and C. T. Long, "Divisibility Properties of Generalized Fibonacci Polynomials," <u>Fibonacci Quarterly</u>, to appear April, 1974.
5. W. A. Webb and E. A. Parberry, "Divisibility Properties of Fibonacci Polynomials," <u>Fibonacci Quarterly</u>, Vol. 7 (Dec. 1969), pp. 457-463.