DIAGONAL SUMS OF THE TRINOMIAL TRIANGLE

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In an earlier paper [1], a method was given for finding the sum of terms along any rising diagonals in any polynomial coefficient array, given by

$$(1 + x + x^2 + \cdots + x^{r-1})^n, \qquad n = 0, 1, 2, \cdots, \quad r \geq 2,$$

which sums generalized the numbers $u(n; p,q)$ of Harris and Styles [2], [3]. In this paper, an explicit solution of the general case for the trinomial triangle is derived.

If we write only the coefficients appearing in the expansions of the trinomial $(1 + x + x^2)^n$, we have

1										
1	1	1								
1	2	3	2	1						
1	3	6	7	6	3	1				
1	4	10	16	19	16	10	4	1		
1	5	15	30	45	51	45	30	15	5	1
					...					

Call the top row the zeroth row and the left-most column the zeroth column. Then, the column generating functions are

$$G_0 = \frac{1}{1-x}, \qquad G_1 = \frac{x}{(1-x)^2}, \qquad G_2 = \frac{x}{(1-x)^3},$$

(1) $$G_{n+2} = \frac{x}{1-x}(G_{n+1} + G_n), \qquad n \geq 0.$$

We desire to find the sums $u(n; p,q)$ which are the sums of those elements found by beginning in the zeroth column and the n^{th} row and taking steps p units up and q units right throughout the left-justified trinomial triangle. Let

(2) $$G = \sum_{n=0}^{\infty} x^{np} G_{nq} = \sum_{n=0}^{\infty} u(n; p,q) x^n .$$

Our first problem is to find a recurrence for every q^{th} column generator. We need two sequences,

$$P_n(x) = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \qquad Q_n(x) = \alpha^n + \beta^n \quad . \tag{3}$$

Both $P_n(x)$ and $Q_n(x)$ obey

$$u_{n+2}(x) = \frac{x}{1-x}(u_{n+1}(x) + u_n(x)) \ .$$

So let

$$A = \frac{x}{1-x};$$

then

$$\begin{aligned} P_{n+2}(x) &= A(P_{n+1}(x) + P_n(x)) \\ Q_{n+2}(x) &= A(Q_{n+1}(x) + Q_n(x)) \\ \alpha^{n+2} &= A(\alpha^{n+1} + \alpha^n) \\ \beta^{n+2} &= A(\beta^{n+1} + \beta^n) \end{aligned} \tag{4}$$

Next, we list the first few members of $P_n(x)$ and $Q_n(x)$.

n	$P_n(x)$	$Q_n(x)$
0	0	2
1	1	A
2	A	$A^2 + 2A$
3	$A^2 + A$	$A^3 + 3A^2$
4	$A^3 + 2A^2$	$A^4 + 4A^3 + 2A^2$
5	$A^4 + 3A^3 + A^2$	$A^5 + 5A^4 + 5A^3$
6	$A^5 + 4A^4 + 3A^3$	$A^6 + 6A^5 + 9A^4 + 2A^3$
...	...	...

Note that the coefficients of $Q_n(x)$ are simply the terms appearing on rising diagonals of the Lucas triangle [4]. The coefficients of $P_n(x)$ and $Q_n(x)$ are the same as those of the Fibonacci and Lucas polynomials, and $P_n(1) = F_n$, $Q_n(1) = L_n$, the n^{th} Fibonacci and Lucas number, respectively.

By mathematical induction, it is easy to show that

$$Q_n(x) = P_{n+1}(x) + AP_{n-1}(x) \quad . \tag{5}$$

Then, the general recurrence for the k^{th} terms is

(6) $$u_{k(n+2)}(x) = Q_k(x)\,u_{k(n+1)}(x) + (-1)^{k+1}A^k u_{kn}(x) .$$

Then, a recurrence relation for every q^{th} column generator is

(7) $$G_{q(n+2)}(x) = Q_q(x)G_{q(n+1)}(x) + (-1)^{q+1}A^q G_{qn}(x) .$$

In summing elements to find $u(n;\ p,q)$ from the column generators, we need to multiply the column generators by powers of x so that the coefficients summed lie along the chosen diagonals of the trinomial array. Then

(8) $$G^*_{q(n+2)}(x) = x^p Q_q(x)G^*_{q(n+1)}(x) + x^{2p}(-1)^{q+1}A^q G^*_{qn}(x) ,$$

$$G^*_0(x) = \frac{1}{1-x} , \qquad G^*_q(x) = x^p G_q(x)$$

Let

$$G_n = \sum_{i=0}^{n} G^*_{iq}$$

and

$$\lim_{n\to\infty} G_n = G ,$$

the generating function for the numbers $u(n;\ p,q)$. We next sum Eq. (8),

$$\sum_{i=0}^{n} G^*_{q(i+2)}(x) = \sum_{i=0}^{n} x^p Q_q(x)\,G^*_{q(i+1)}(x) + \sum_{i=0}^{n} x^{2p}(-1)^{q+1}A^q G^*_{qi}(x) ,$$

which becomes, upon expansion,

$$\begin{aligned} G_n - G^*_q(x) - G^*_0(x) + G^*_{(n+1)q}(x) + G^*_{(n+2)q}(x) \\ = x^p Q_q(x)G^*_{(n+1)q}(x) + x^p Q_q(x)G_n - x^p Q_q(x)G^*_0(x) \\ + x^{2p}(-1)^{q+1}A^q G_n . \end{aligned}$$

Collecting terms, our sum simplifies to

$$G_n(1 - x^p Q_q(x) - x^{2p}(-1)^{q+1}A^q) = G^*_0(x)\,(1 - x^p Q_q(x)) + G^*_q(x) + R_n ,$$

where R_n involves only terms involving $G^*_{(n+1)q}(x)$ and $G^*_{(n+2)q}(x)$. It can be shown that

$$\lim_{n\to\infty} G^*_n(x) = 0$$

for $|x| < 1/r$, $r > 2$, so that $\lim_{n \to \infty} R_n = 0$. Then, taking the limit as $n \to \infty$ of our sum and simplifying,

$$G = \frac{G_0^*(x)(1 - x^p Q_q(x)) + G_q^*(x)}{1 - x^p Q_q(x) + x^{2p}(-A)^q},$$

which becomes Eq. (9) from the identity given in Eq. (8):

$$G = \frac{G_0(x)(1 - x^p Q_q(x)) + x^p G_q(x)}{1 - x^p Q_q(x) + x^{2p}(-A)^q} = \sum_{n=0}^{\infty} u(n;\, p,q) x^n \quad, \tag{9}$$

where $G_n(x)$ is defined by Eq. (1), $A = \dfrac{x}{1 - x}$, and

$$Q_k(x) = \sum_{i=0}^{[(k+1)/2]} \left[\binom{k - i}{i} + \binom{k - i - 1}{i - 1} \right] A^{k-i} \,. \tag{10}$$

REFERENCES

1. V. E Hoggatt, Jr., and Marjorie Bicknell, "Diagonal Sums of Generalized Pascal Triangles," Fibonacci Quarterly, Vol. 7, No. 4, Nov. 1969, pp. 341-358.
2. V. C. Harris and Carolyn C. Styles, "A Generalization of Fibonacci Numbers," Fibonacci Quarterly, Vol. 2, No. 4, Dec. 1964, pp. 277-289.
3. V. C. Harris and Carolyn C. Styles, "Generalized Fibonacci Sequences Associated with a Generalized Pascal Triangle," Fibonacci Quarterly, Vol. 4, No. 3, Oct. 1966, pp. 241-248.
4. Verner E. Hoggatt, Jr., "An Application of the Lucas Triangle," Fibonacci Quarterly, Vol. 8, No. 4, Oct. 1970, pp. 360-364.