SOME GEOMETRICAL PROPERTIES OF THE GENERALIZED FIBONACCI SEQUENCE

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1. INTRODUCTION

In this paper, some geometrical properties of the generalized Fibonacci sequence $\{T_n\}$ have been discussed. The sequence $\{T_n\}$ being defined by

$$T_{n+1} = T_n + T_{n-1},$$
$$T_1 = a, \quad T_2 = b.$$

On taking $a = b = 1$, the Fibonacci sequence $\{F_n\}$ is obtained.

We shall make use of the following identities [1]

$$T_{m+n} = T_m F_{n+1} + T_{m-1} F_n. \tag{1.1}$$

$$F_n F_{n+m} - F_{n-s} F_{n+m+s} = (-1)^{n-s} F_s F_{s+m}. \tag{1.2}$$

$$T_m T_{n+k} - T_{m+k} T_n = (-1)^m F_k F_{n-m} D, \tag{1.3}$$

where D is the characteristic number of the sequence and is given by

$$T_n^2 - T_{n-1} T_{n+1} = (-1)^n D; \qquad 2a < b.$$

2. THEOREM 1

Area of the triangle having vertices at the points designated by the rectangular cartesian coordinates (T_n, T_{n+r}), (T_{n+p}, T_{n+p+r}), (T_{n+q}, T_{n+q+r}) is independent of n.

Proof. Twice the area of the specified triangle is equal to the absolute value of the determinant

$$\begin{vmatrix} T_n & T_{n+r} & 1 \\ T_{n+p} & T_{n+p+r} & 1 \\ T_{n+q} & T_{n+q+r} & 1 \end{vmatrix}.$$

Using (1.1) for the second column the determinant can be written as

$$F_{r+1}\begin{vmatrix} T_n & T_n & 1 \\ T_{n+p} & T_{n+p} & 1 \\ T_{n+q} & T_{n+q} & 1 \end{vmatrix} + F_r \begin{vmatrix} T_n & T_{n-1} & 1 \\ T_{n+p} & T_{n+p-1} & 1 \\ T_{n+q} & T_{n+q-1} & 1 \end{vmatrix}.$$

The first determinant is obviously zero; in the second on alternately subtracting the second and first column from each other, the suffixes can be reduced and finally we get

$$\pm F_r \begin{vmatrix} T_1 & T_2 & 1 \\ T_{p+1} & T_{p+2} & 1 \\ T_{q+1} & T_{q+2} & 1 \end{vmatrix}$$

according as n is odd or even.

On expanding the determinant along the third column, we obtain

$$\pm F_r[(T_{p+1}T_{q+2} - T_{p+2}T_{q+1}) - (T_1T_{q+2} - T_2T_{q+1}) + (T_1T_{p+2} - T_2T_{p+1})] \quad ,$$

which on using (1.3) reduces to

$$\pm F_r[F_q - F_p - (-1)^p F_{q-p}]D \quad .$$

Thus the area of the specified triangle is independent of n.

Particular Case. On taking $r = h$, $p = 2h$, $q = 4h$, $a = b = 1$, we find that the area of the triangle whose vertices are (F_n, F_{n+h}), (F_{n+2h}, F_{n+3h}), (F_{n+4h}, F_{n+5h}) is equal to the value of

$$\tfrac{1}{2}F_h(F_{4h} - 2F_{2h}) \quad . \tag{2.1}$$

Duncan [2] has proved that the area of this triangle is

$$\tfrac{1}{2}[F_h(F_{4h} - F_{2h}) - (F_{3h}F_{4h} - F_{2h}F_{5h})] \quad ,$$

which on using (1.2) simplifies to the value given in (2.1).

3. THEOREM 2

Lines drawn through the origin with the direction ratios T_n, T_{n+p}, T_{n+q}, where p and q are arbitrary constants are always coplanar for every value of n.

Proof. Direction ratios of any three such lines are T_i, T_{i+p}, T_{i+q}; T_j, T_{j+p}, T_{j+q}; T_k, T_{k+p}, T_{k+q}. These will be coplanar if

$$\begin{vmatrix} T_i & T_{i+p} & T_{i+q} \\ T_j & T_{j+p} & T_{j+q} \\ T_k & T_{k+p} & T_{k+q} \end{vmatrix} = 0 \quad . \tag{3.1}$$

On using the relation (1.1), the left-hand side of (3.1) can be written as the sum of four determinants, each of which is zero. Hence proved.

4. THEOREM 3

Set of points designated by the cartesian coordinates (T_n, T_{n+p}, T_{n+q}) where p and

q are arbitrary constants and $n = 1, 2, 3, \cdots$, are always coplanar. This plane passes through the origin, and its equation is independent of n.

Proof. Equation to the plane passing through any three points of the set is

$$\begin{vmatrix} x & y & z & 1 \\ T_i & T_{i+p} & T_{i+q} & 1 \\ T_j & T_{j+p} & T_{j+q} & 1 \\ T_k & T_{k+p} & T_{k+q} & 1 \end{vmatrix} = 0 , \tag{4.1}$$

where i, j and k are particular values of n. Here the coefficient of x is

$$\begin{aligned} &= [(T_{j+p}T_{k+q} - T_{j+q}T_{k+p}) - (T_{i+p}T_{k+q} - T_{i+q}T_{k+p}) \\ &\qquad + (T_{i+p}T_{j+q} - T_{i+q}T_{j+p})] \\ &= (-1)^p F_{q-p}\{(-1)^j F_{k-j} - (-1)^i F_{k-i} + (-1)^i F_{j-i}\} D . \end{aligned}$$

The coefficient of y is obtained on putting $p = 0$ in the coefficient of x; the coefficient of z is obtained from the coefficient of y on replacing q by p; the constant term is zero as is already proved in (3.1).

Thus the equation to the plane simplifies to

$$(-1)^p F_{q-p}x - F_q y + F_p z = 0 . \tag{4.2}$$

This equation is independent of n. Also it does not depend on the initial values a and b. Q.E.D.

Particular Case. On taking $a = 1$, $b = 3$ we obtain the Lucas sequence $\{L_n\}$. The points (F_i, F_{i+2}, F_{i+5}), $i = 1, 2, 3, \cdots$; (L_j, L_{j+2}, L_{j+5}), $j = 1, 2, 3, \cdots$; (T_k, T_{k+2}, T_{k+5}), $k = 1, 2, 3, \cdots$; all lie on the plane $2x - 5y + z = 0$.

5. THEOREM 4

The set of planes

$$T_n x + T_{n+p} y + T_{n+q} z + T_{n+r} = 0 ,$$

where p, q, r are arbitrary constants, and $n = 1, 2, 3, \cdots$; all intersect in a given line whose equation is independent of n.

Proof. Let two such planes be

$$\begin{aligned} T_i x + T_{i+p} y + T_{i+q} z + T_{i+r} &= 0 \\ T_j x + T_{j+p} y + T_{j+q} z + T_{j+r} &= 0 . \end{aligned} \tag{5.1}$$

The equation to the line of intersection of the parallel planes through the origin is

$$\frac{x}{T_{i+p}T_{j+q} - T_{i+q}T_{j+p}} = \frac{y}{T_i T_{j+q} - T_{i+q}T_j} = \frac{z}{T_i T_{j+p} - T_{i+p}T_j}$$

On using (1.3) and proceeding as in (4.2) this simplifies to

$$\frac{x}{(-1)^p F_{q-p}} = \frac{-y}{F_q} = \frac{z}{F_p} .$$

Similarly the line of intersection of the planes given by (5.1) meets the plane $z = 0$, at the point given by

$$\frac{x}{(-1)^p F_{r-p}} = \frac{-y}{F_r} = \frac{1}{F_p} .$$

Thus the equation to the line of intersection of the planes given by (5.1) becomes

$$\frac{(-1)^p F_p x - F_{r-p}}{F_{q-p}} = \frac{F_p y + F_r}{-F_q} = \frac{z}{F_p} . \tag{5.2}$$

Hence proved.

Particular Case. The set of planes whose equations are

$$F_i x + F_{i+1} y + F_{i+3} z + F_{i+4} = 0, \quad i = 1, 2, 3, \cdots ;$$
$$L_j x + L_{j+1} y + L_{j+3} z + L_{j+4} = 0, \quad j = 1, 2, 3, \cdots ;$$
$$T_k x + T_{k+1} y + T_{k+3} z + T_{k+4} = 0, \quad k = 1, 2, 3, \cdots ;$$

all intersect along the line

$$\frac{x+2}{1} = \frac{y+3}{2} = \frac{z}{-1} .$$

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2. Dewey C. Duncan, "Chains of Equivalent Fibonacci-wise Triangles," Fibonacci Quarterly, Vol. 5, No. 1 (February 1967), pp. 87-88.