ITERATION ALGORITHMS FOR CERTAIN SUMS OF SQUARES

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The following three-step iteration algorithm to generate \(x\) simultaneously in \(2x + 1 = a^2\) and \(3x + 1 = b^2\) was mentioned, but not proved, in [4, p. 211]:

\[
\begin{align*}
1 \cdot 10 - 1 & = 9 \quad 9^2 = 81 \quad (81 - 1)/2 = 40 = x_1 \\
9 \cdot 10 - 1 & = 89 \quad 89^2 = 7921 \quad (7921 - 1)/2 = 3960 = x_2 \\
89 \cdot 10 - 9 & = 881 \quad 881^2 = 776161 \quad (776161 - 1)/2 = 388080 = x_3 \\
881 \cdot 10 - 89 & = 8721 \quad 8721^2 = 76055841 \quad (76055841 - 1)/2 = 38027920 = x_4 \\
8721 \cdot 10 - 881 & = 86329 \quad 86329^2 = 7452696241 \quad (7452696241 - 1)/2 = 3726348120 = x_5.
\end{align*}
\]

**Proof.** From \(2x + 1 = a^2\) and \(3x + 1 = b^2\) follows \(3a^2 - 2b^2 = 1\). If \(a_n, b_n\) is any solution of this generalized Pell equation, then \(a_{n+1} = 5a_n + 4b_n\), \(b_{n+1} = 6a_n + 5b_n\) is the next larger one. From these, we obtain immediately \(a_{n+1} + a_{n-1} = 10a_n\), \(b_{n+1} + b_{n-1} = 10b_n\), which is equivalent to the algorithm.

For the \(n\)th term formula we use the usual approach by linear substitutions (for example, [1, p. 181]) and obtain

\[
x_n = \left[\left(\sqrt{5} + 2\right)\left(5 + 2\sqrt{5}\right)^n + \left(\sqrt{5} - 2\right)\left(5 - 2\sqrt{5}\right)^n\right]/48 - 1/2.
\]

This formula has three shortcomings: (1) it uses fractions, (2) it employs roots, and (3) it has \(n\) in the exponent. The algorithm above has none of them.

Similar arguments are valid for a four-step iteration algorithm [3] to generate \(x\) in \(x^2 + (x + 1)^2 = y^2\).

Sometimes, the \(n\)th term formula may be simple, as for \(a^2 + b^2 + (ab)^2 = c^2\), \(a\) and \(b\) consecutive positive integers [2]. Here we have

\[
(n - 1)^2 + n^2 + [ (n - 1)n] ^2 = (n^2 - n + 1)^2,
\]

and hence we need no algorithm. But for \(a = 1\) an algorithm would be helpful. Let us first find some clues to such an algorithm. We have by hand and by a table of squares:

\[
\begin{align*}
1^2 + 0^2 + 0^2 & = 1^2 = (0^2 + 1)^2 \\
1^2 + 1^2 + 1^2 & = 3^2 = (2^2 - 1)^2 \\
1^2 + 2^2 + 2^2 & = 3^2 = (2^2 - 1)^2 \\
1^2 + 12^2 + 12^2 & = 17^2 = (4^2 + 1)^2 \\
1^2 + 70^2 + 70^2 & = 99^2 = (10^2 - 1)^2.
\end{align*}
\]
The alternating +1 and -1 in the last column, which shows a constant pattern, suggests the possibility of an algorithm. If we can find all \( b \), say, from \( b_3 = 12 \) on, we will also have all \( c \). After some trials and errors, we obtain

**Iteration Algorithm 1**

\[
\begin{align*}
6 \cdot 2 - 0 & = 12 \\
6 \cdot 12 - 2 & = 70 \\
6 \cdot 70 - 12 & = 408 \\
6 \cdot 408 - 70 & = 2378 \\
6 \cdot 2378 - 408 & = 13860 \\
6 \cdot 13860 - 2378 & = 80782
\end{align*}
\]

which yields easily the next four results:

\[
\begin{align*}
1^2 + 408^2 + 408^2 & = 577^2 = (24^2 + 1)^2 \\
1^2 + 2378^2 + 2378^2 & = 3363^2 = (58^2 - 1)^2 \\
1^2 + 13860^2 + 13860^2 & = 19601^2 = (140^2 + 1)^2 \\
1^2 + 80782^2 + 80782^2 & = 114243^2 = (338^2 - 1)^2
\end{align*}
\]

Similarly, we approach the case \( a = 2 \). We have by hand and a table of squares:

\[
\begin{align*}
2^2 + 1^2 + 2^2 & = 3^2 = (1^2 + 2)^2 \\
2^2 + 3^2 + 6^2 & = 7^2 = (3^2 - 2)^2 \\
2^2 + 8^2 + 16^2 & = 18^2 = (4^2 + 2)^2 \\
2^2 + 21^2 + 42^2 & = 47^2 = (7^2 - 2)^2
\end{align*}
\]

The alternating +2 and -2 in the last column, which shows a constant pattern, suggests the possibility of an algorithm. If we can find all \( b \), say, from \( b_3 = 8 \) on, we will also have all \( c \). After some trials and errors we obtain:

**Iteration Algorithm 2**

\[
\begin{align*}
3 \cdot 3 - 1 & = 8 \\
3 \cdot 8 - 3 & = 21 \\
3 \cdot 21 - 8 & = 55 \\
3 \cdot 55 - 21 & = 144 \\
3 \cdot 144 - 55 & = 377 \\
3 \cdot 377 - 144 & = 987
\end{align*}
\]

which yields easily the next four results:
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\[ 2^2 + 55^2 + 110^2 = 123^2 = (11^2 + 2)^2 \]
\[ 2^2 + 144^2 + 285^2 = 322^2 = (18^2 - 2)^2 \]
\[ 2^2 + 377^2 + 754^2 = 843^2 = (29^2 + 2)^2 \]
\[ 2^2 + 987^2 + 1974^2 = 2207^2 = (47^2 - 2)^2 . \]

Slightly different behaves the case \( a = 3 \). We have by hand and a table of squares:

\[ 3^2 + 0^2 + 0^2 = 3^2 = (0^2 + 3)^2 \]
\[ 3^2 + 2^2 + 6^2 = 7^2 = (2^2 + 3)^2 \]
\[ 3^2 + 4^2 + 12^2 = 13^2 = (4^2 - 3)^2 \]
\[ 3^2 + 18^2 + 54^2 = 57^2 \]
\[ 3^2 + 86^2 + 240^2 = 253^2 = (16^2 - 3)^2 \]
\[ 3^2 + 154^2 + 462^2 = 487^2 = (22^2 + 3)^2 \]
\[ 3^2 + 684^2 + 2052^2 = 2163^2 . \]

Here the doubly alternating \(+3\) and \(-3\) in the last column would show a constant pattern, if the exceptional values 57\(^2\) and 2163\(^2\) could be eliminated. This suggests the possibility of two algorithms. To obtain further results, we write an Integer-FORTRAN program for the IBM 1130 which yields

\[ 3^2 + 3038^2 + 9114^2 = 9607^2 = (98^2 + 3)^2 \]
\[ 3^2 + 5848^2 + 17544^2 = 18493^2 = (136^2 - 3)^2 \]
\[ 3^2 + 25974^2 + 77922^2 = 82137^2 \]
\[ 3^2 + 115364^2 + 346092^2 = 364813^2 = (604^2 - 3)^2 \]
\[ 3^2 + 222070^2 + 666210^2 = 702247^2 = (838^2 + 3)^2 \]
\[ 3^2 + 983238^2 + 2958984^2 = 3119043^2 \]
\[ 3^2 + 4380794^2 + 13142382^2 = 13853287^2 = (3722^2 + 3)^2 . \]

Now we want to find an algorithm which should generate the sequence 80, 154, 3038, 5848, 115364, 222070, 4380794, \ldots. Let the terms \( b_1 = 0 \), \( b_2 = 2 \), and \( b_3 = 4 \) be given; then \( b_0 = -4 \) is the left neighbor of \( b_1 = 0 \), since \( 3^2 + (-4)^2 + (-12)^2 = 13^2 = (4^2 - 3)^2 \) is the logical extension to the left. With this trick and some trials and errors, we obtain

**Iteration Algorithm 3**

\[ 38 \cdot 2 - (-4) = 80 \]
\[ 2 \cdot 80 - 2 \cdot 4 + 2 = 154 \]
\[ 38 \cdot 80 - 2 = 3038 \]
\[ 2 \cdot 3038 - 2 \cdot 154 + 80 = 5848 \]
\[ 38 \cdot 3038 - 80 = 115364 \]
\[ 2 \cdot 115364 - 2 \cdot 5848 + 3038 = 222070 \]
\[ 38 \cdot 115364 - 3038 = 4380794 . \]
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Now there remains only to find an algorithm which should generate \( 25974, 986328, \ldots \). Here we have not far to go, since such an algorithm is already contained in the former one, and we obtain easily

\[
\begin{align*}
\text{Iteration Algorithm 4} \\
38\cdot 684 - 18 &= 25974 \\
38\cdot 25974 - 684 &= 986328.
\end{align*}
\]

Finally, one could ask: Does there exist a general formula for solving \( x^2 + y^2 + z^2 = w^2 \)? The answer is yes. Let \( x = p^2 + q^2 - r^2 \), \( y = 2pr \), \( z = 2qr \), and \( w = p^2 + q^2 + r^2 \); then \( x^2 + y^2 + z^2 = w^2 \) becomes \( 0 = 0 \). But this formula has two shortcomings: (1) it uses fractions, and (2) it employs roots, since, for example, the solution of \( 3^2 + 2^2 + 6^2 = 7^2 \) requires \( p = \sqrt{3}/2 \), \( q = 3\sqrt{2}/2 \), and \( r = \sqrt{2} \).

REFERENCES


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