ITERATION ALGORITHMS FOR CERTAIN SUMS OF SQUARES

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The following three-step iteration algorithm to generate x simultaneously in $2x + 1 = a^2$ and $3x + 1 = b^2$ was mentioned, but not proved, in [4, p. 211]:

$$
\begin{array}{rclrclrcl}
1 \cdot 10 - 1 &=& 9 & 9^2 &=& 81 & (81 - 1)/2 &=& 40 = x_1 \\
9 \cdot 10 - 1 &=& 89 & 89^2 &=& 7921 & (7921 - 1)/2 &=& 3960 = x_2 \\
89 \cdot 10 - 9 &=& 881 & 881^2 &=& 776161 & (776161 - 1)/2 &=& 388080 = x_3 \\
881 \cdot 10 - 89 &=& 8721 & 8721^2 &=& 76055841 & (76055841 - 1)/2 &=& 38027920 = x_4 \\
8721 \cdot 10 - 881 &=& 86329 & 86329^2 &=& 7452696241 & (7452696241 - 1)/2 &=& 3726348120 = x_5 .
\end{array}
$$

Proof. From $2x + 1 = a^2$ and $3x + 1 = b^2$ comes $3a^2 - 2b^2 = 1$. If a_n, b_n is any solution of this generalized Pell equation, then $a_{n+1} = 5a_n + 4b_n$, $b_{n+1} = 6a_n + 5b_n$ is the next larger one. From these, we obtain immediately $a_{n+1} + a_{n-1} = 10a_n$, $b_{n+1} + b_{n-1} = 10b_n$, which is equivalent to the algorithm.

For the n^{th} term formula we use the usual approach by linear substitutions (for example, [1, p. 181]) and obtain

$$x_n = [(\sqrt{6} + 2)(5 + 2\sqrt{6})^n + (\sqrt{6} - 2)(5 - 2\sqrt{6})^n]^2/48 - 1/2 .$$

This formula has three shortcomings: (1) it uses fractions, (2) it employs roots, and (3) it has n in the exponent. The algorithm above has none of them.

Similar arguments are valid for a four-step iteration algorithm [3] to generate x in $x^2 + (x + 1)^2 = y^2$.

Sometimes, the n^{th} term formula may be simple, as for $a^2 + b^2 + (ab)^2 = c^2$, a and b consecutive positive integers [2]. Here we have

$$(n - 1)^2 + n^2 + [(n - 1)n]^2 = (n^2 - n + 1)^2 ,$$

and hence we need no algorithm. But for $a = 1$ an algorithm would be helpful. Let us first find some clues to such an algorithm. We have by hand and by a table of squares:

$$
\begin{array}{l}
1^2 + 0^2 + 0^2 = 1^2 = (0^2 + 1)^2 \\
1^2 + 2^2 + 2^2 = 3^2 = (2^2 - 1)^2 \\
1^2 + 12^2 + 12^2 = 17^2 = (4^2 + 1)^2 \\
1^2 + 70^2 + 70^2 = 99^2 = (10^2 - 1)^2 .
\end{array}
$$

The alternating +1 and −1 in the last column, which shows a constant pattern, suggests the possibility of an algorithm. If we can find all b, say, from $b_3 = 12$ on, we will also have all c. After some trials and errors, we obtain

Iteration Algorithm 1

$$
\begin{aligned}
6\cdot 2 - 0 &= 12\\
6\cdot 12 - 2 &= 70\\
6\cdot 70 - 12 &= 408\\
6\cdot 408 - 70 &= 2378\\
6\cdot 2378 - 408 &= 13860\\
6\cdot 13860 - 2378 &= 80782
\end{aligned}
$$

which yields easily the next four results:

$$
\begin{aligned}
1^2 + 408^2 + 408^2 &= 577^2 = (24^2 + 1)^2\\
1^2 + 2378^2 + 2378^2 &= 3363^2 = (58^2 - 1)^2\\
1^2 + 13860^2 + 13860^2 &= 19601^2 = (140^2 + 1)^2\\
1^2 + 80782^2 + 80782^2 &= 114243^2 = (338^2 - 1)^2 .
\end{aligned}
$$

Similarly, we approach the case $a = 2$, We have by hand and a table of squares:

$$
\begin{aligned}
2^2 + 1^2 + 2^2 &= 3^2 = (1^2 + 2)^2\\
2^2 + 3^2 + 6^2 &= 7^2 = (3^2 - 2)^2\\
2^2 + 8^2 + 16^2 &= 18^2 = (4^2 + 2)^2\\
2^2 + 21^2 + 42^2 &= 47^2 = (7^2 - 2)^2 .
\end{aligned}
$$

The alternating +2 and −2 in the last column, which shows a constant pattern, suggests the possibility of an algorithm. If we can find all b, say, from $b_3 = 8$ on, we will also have all c. After some trials and errors we obtain:

Iteration Algorithm 2

$$
\begin{aligned}
3\cdot 3 - 1 &= 8\\
3\cdot 8 - 3 &= 21\\
3\cdot 21 - 8 &= 55\\
3\cdot 55 - 21 &= 144\\
3\cdot 144 - 55 &= 377\\
3\cdot 377 - 144 &= 987
\end{aligned}
$$

which yields easily the next four results:

$$2^2 + 55^2 + 110^2 = 123^2 = (11^2 + 2)^2$$
$$2^2 + 144^2 + 288^2 = 322^2 = (18^2 - 2)^2$$
$$2^2 + 377^2 + 754^2 = 843^2 = (29^2 + 2)^2$$
$$2^2 + 987^2 + 1974^2 = 2207^2 = (47^2 - 2)^2 .$$

Slightly different behaves the case $a = 3$. We have by hand and a table of squares:

$$3^2 + 0^2 + 0^2 = 3^2 = (0^2 + 3)^2$$
$$3^2 + 2^2 + 6^2 = 7^2 = (2^2 + 3)^2$$
$$3^2 + 4^2 + 12^2 = 13^2 = (4^2 - 3)^2$$
$$3^2 + 18^2 + 54^2 = 57^2$$
$$3^2 + 80^2 + 240^2 = 253^2 = (16^2 - 3)^2$$
$$3^2 + 154^2 + 462^2 = 487^2 = (22^2 + 3)^2$$
$$3^2 + 684^2 + 2052^2 = 2163^2 .$$

Here the doubly alternating $+3$ and -3 in the last column would show a constant pattern, if the exceptional values 57^2 and 2163^2 could be eliminated. This suggests the possibility of two algorithms. To obtain further results, we write an Integer-FORTRAN program for the IBM 1130 which yields

$$3^2 + 3038^2 + 9114^2 = 9607^2 = (98^2 + 3)^2$$
$$3^2 + 5848^2 + 17544^2 = 18493^2 = (136^2 - 3)^2$$
$$3^2 + 25974^2 + 77922^2 = 82137^2$$
$$3^2 + 115364^2 + 346092^2 = 364813^2 = (604^2 - 3)^2$$
$$3^2 + 222070^2 + 666210^2 = 702247^2 = (838^2 + 3)^2$$
$$3^2 + 986328^2 + 2958984^2 = 3119043^2$$
$$3^2 + 4380794^2 + 13142382^2 = 13853287^2 = (3722^2 + 3)^2 .$$

Now we want to find an algorithm which should generate the sequence 80, 154, 3038, 5848, 115364, 222070, 4380794, $\cdots$. Let the terms $b_1 = 0$, $b_2 = 2$, and $b_3 = 4$ be given; then $b_0 = -4$ is the left neighbor of $b_1 = 0$, since $3^2 + (-4)^2 + (-12)^2 = 13^2 = (4^2 - 3)^2$ is the logical extension to the left. With this trick and some trials and errors, we obtain

Iteration Algorithm 3

$$38 \cdot 2 - (-4) = 80$$
$$2 \cdot 80 - 2 \cdot 4 + 2 = 154$$
$$38 \cdot 80 - 2 = 3038$$
$$2 \cdot 3038 - 2 \cdot 154 + 80 = 5848$$
$$38 \cdot 3038 - 80 = 115364$$
$$2 \cdot 115364 - 2 \cdot 5848 + 3038 = 222070$$
$$38 \cdot 115364 - 3038 = 4380794 .$$

Now there remains only to find an algorithm which should generate 25974, 986328, $\cdots$. Here we have not far to go, since such an algorithm is already contained in the former one, and we obtain easily

Iteration Algorithm 4

$$38 \cdot 684 - 18 = 25974$$

$$38 \cdot 25974 - 684 = 986328 \quad .$$

Finally, one could ask: Does there exist a general formula for solving $x^2 + y^2 + z^2 = w^2$? The answer is yes. Let $x = p^2 + q^2 - r^2$, $y = 2pr$, $z = 2qr$, and $w = p^2 + q^2 + r^2$; then $x^2 + y^2 + z^2 = w^2$ becomes $0 = 0$. But this formula has two shortcomings: (1) it uses fractions, and (2) it employs roots, since, for example, the solution of $3^2 + 2^2 + 6^2 = 7^2$ requires $p = \sqrt{2}/2$, $q = 3\sqrt{2}/2$, and $r = \sqrt{2}$.

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4. Zentralblatt für Mathematik, 151 (1968), pp. 211-212.