ADVANCED PROBLEMS AND SOLUTIONS

Edited by
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Send all communications concerning Advanced Problems and Solutions to Raymond E. Whitney, Mathematics Department, Lock Haven State College, Lock Haven, Pennsylvania 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, solutions should be submitted on separate signed sheets within two months after publication of the problems.


Suppose an alphabet, \( A = \{x_1, x_2, x_3, \ldots\} \), is given along with a binary connective, \( P \) (in prefix form). Define a well formed formula (wff) as follows: a wff is

1. \( x_i \) for \( i = 1, 2, 3, \ldots \), or
2. If \( A_1, A_2 \) are wff’s, then \( PA_1A_2 \) is a wff and
3. The only wff’s are of the above two types.

A wff of order \( n \) is a wff in which the only alphabet symbols are \( x_1, x_2, \ldots, x_n \) in that order with each letter occurring exactly once. There is one wff of order 1, namely \( x_1 \). There is one wff of order 2, namely \( Px_1x_2 \). There are two wff’s of order 3, namely \( Px_1Px_2x_3 \) and \( PPx_1x_2x_3 \), and there are five wff’s of order 4, etc.

Define a sequence

\[ \{ G_i \}_{i=1}^{\infty} \]

as follows:

\( g_i \) is the number of distinct wff’s of order \( i \).

a. Find a recurrence relation for \( \{ G_i \}_{i=1}^{\infty} \) and

b. Find a generating function for \( \{ G_i \}_{i=1}^{\infty} \).
a. Find the second-order ordinary differential equation whose power series solution is

\[ \sum_{n=0}^{\infty} F_{n+1} x^n. \]

b. Find the second-order ordinary differential equation whose power series solution is

\[ \sum_{n=0}^{\infty} I_{n+1} x^n. \]

**H-236 Proposed by L. Carlitz, Duke University, Durham, North Carolina.**

Show that

\[
\sum_{n=0}^{\infty} (-1)^n x^n = \sum_{n=0}^{\infty} \frac{x^{2n}}{(x)_{2n}} \prod_{k=1}^{\infty} (1 - x^k),
\]

\[
\sum_{n=0}^{\infty} (-1)^n (n+1)^2 x^n = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(x)_{2n+1}} \prod_{k=1}^{\infty} (1 - x^k),
\]

where \((x)_k = (1 - x)(1 - x^3) \cdots (1 - x^k), \quad (x)_0 = 1.\]

**SOLUTIONS**

**TO COIN A THEOREM**

**H-199 Proposed by L. Carlitz and R. Scoville, Duke University, Durham, North Carolina.**

A certain country's coinage consists of an infinite number of types of coins: \( \cdots, \ C_{-2}, \ C_{-1}, \ C_0, \ C_1, \cdots \). The value \( V_n \) of the coin \( C_n \) is related to the others as follows: for all \( n, \)

\[ V_n = V_{n-3} + V_{n-2} + V_{n-1}. \]

Show that any (finite) pocketful of coins is equal in value to a pocketful containing at most one coin of each type.
Solution by the Proposer.

Call a pocketful \( Q \) canonical if it consists entirely of coins of different types and such that no three coins of "adjacent" types (e.g., \( C_n, C_{n+1} \) and \( C_{n+2} \)) are present. Call two pocketfuls equivalent if they have the same values.

We will prove for any pocketful \( P \) the statement:

\[
S: \ P \text{ is equivalent to a canonical pocketful.}
\]

Note that any pocketful containing only differing types is equivalent to a canonical pocketful since the three adjacent coins of highest value, \( C_n, C_{n+1}, C_{n+2} \) can be replaced by \( C_{n+3} \), etc.

Assume for the moment, the following statement:

\[
R: \ S \text{ is true for any canonical pocketful to which one extra coin has been added.}
\]

Then the general result follows by induction on the number of coins for any pocketful \( P \): Remove a coin to get \( P' \), apply the induction hypothesis to \( P' \) to get a canonical pocketful \( Q \), return the removed coin and apply \( R \).

Now to prove \( R \), let us prove by induction on \( j \) the series of statements \( R_j \): 

\[
R_j \begin{cases}
\text{If } Q \text{ is any canonical pocketful in which the coin of least value is a } C_n, \text{ then if a } C_{n+j} \text{ or a } C_n \text{ and a } C_{n+j+1} \text{ be added to } Q \text{ to get a pocketful } P', \text{ then } S \text{ is true for } P' .
\end{cases}
\]

Assume \( R_k \) for all \( k < j \) (it is obvious if \( k \leq -3 \)). Now let \( Q \) be canonical. We can suppose that \( n + j = 0 \). Suppose \( Q \) contains \( \delta_1 \) coins of type \( C_1 \), \( \delta_2 = 0 \) or 1, \( \delta_2 \delta_1 + \delta_1 + \delta_2 = 0 \) for all \( i \). Then

* \[
Q \cup C_0 \equiv \cdots \delta_3, \delta_2, \delta_1, \delta_0 + 1, \delta_1, \delta_2, \delta_3 , \cdots
\]

If \( \delta_0 = 0 \), we are finished, so assume \( \delta_0 = 1 \). Then

\[
Q \cup C_0 \equiv \cdots \delta_3 + 1, \delta_2, \delta_1, 0, \delta_1 + 1, \delta_2, \delta_3 , \cdots
\]

Again, if \( \delta_1 = 0 \), by induction, we are finished, so assume \( \delta_1 = 1 \). Then

\[
Q \cup C_0 \equiv \cdots \delta_3 + 1, \delta_2 + 1, \delta_1, 0, 0, \delta_2 + 1, \delta_3 , \cdots
\]

Now, since \( \delta_0 \delta_2 = 0, \delta_2 = 0 \) so we are finished.

For the next part,

\[
Q \cup C_0 \cup C_1 \equiv \cdots \delta_3, \delta_2, \delta_1, \delta_0 + 1, \delta_1 + 1, \delta_2, \delta_3 , \cdots
\]
If either $\delta_9$ or $\delta_1 = 0$, this case can be handled as above, so suppose $\delta_9$ and $\delta_1$ are

1. Then

$$Q \cup C_9 \cup C_1 \equiv \ldots \delta_3 \delta_2 \delta_1 \ 2 \ 2 \ \delta_2 \ \delta_3 \ldots$$

$$\equiv \ldots \delta_3 + 1 \delta_2 + 1 \delta_1 + 1 \ 1 \ 2 \ \delta_2 \ \delta_3 \ldots$$

$$\equiv \ldots \delta_3 + 1 \delta_2 + 2 \delta_1 + 2 \ 1 \ \delta_2 \ \delta_3 \ldots$$

$$\equiv \ldots \delta_3 + 1 \delta_2 + 1 \delta_1 + 1 \ 0 \ \delta_2 + 1 \ \delta_3 \ldots$$

$$\equiv \ldots \delta_3 + 1 \delta_2 + 1 \delta_1 + 0 \ \delta_2 + 1 \ \delta_3 \ldots$$

and again, by induction, we are finished. This completes the proof.

We note, without proof, that no two canonical pocketfuls are equivalent.

**Editorial Note:** The given sequences identify the elements of the union.

**ASYMPTOTIC PI**

**H-200 Proposed by Guy A. R. Guillotte, Cowansville, Quebec, Canada.**

Let $M(n)$ be the number of primes (distinct) which divide the binomial coefficient,

$$\binom{n}{k}^*.$$}

Clearly, for $1 \leq n \leq 15$, we have $M(1) = 0$, $M(2) = M(3) = 1$, $M(4) = M(5) = 2$, $M(6) = M(7) = M(8) = M(9) = 3$, $M(10) = 4$, $M(11) = M(12) = M(14) = 5$, $M(13) = M(15) = 6$, etc. Show that

$$\left\{m(n)\right\}_{n=1}^\infty$$

has an upper bound and find an asymptotic formula for $M(n)$.

*Divide at least one $\binom{n}{k}$ where $0 \leq k \leq n$.

**Solution by D. Singmaster, Instituto Mathematica, Pisa, Italy.**

For a prime $p$, if

$$p \left| \binom{n}{k} \right.$$
for some \( k, \ 0 \leq k \leq n \), then \( p \mid n! \) and so \( p \leq n \). Hence \( M(n) \leq \pi(n) \), where \( \pi(n) \) is the number of primes less than or equal to \( n \). We claim \( M(n) \sim \pi(n) \). To see this, we can use the following result of B. Ram. (See: L. E. Dickson, History of the Theory of Numbers, Vol. 1; Chelsea, 1952; p. 274, item 98.) There is at most one prime \( p < n \) such that

\[
p \mid \binom{n}{k}
\]

for \( 0 \leq k \leq n \) and such a prime \( p \) exists if and only if \( n + 1 = a p^s \) with \( 1 \leq a < p < n \).

Since Ram's paper is somewhat inaccessible, I will prove a slight sharpening of it, using an accessible result. N. J. Fine ("Binomial Coefficients modulo a Prime," Amer. Math. Monthly, 54 (1947), 589-592, Theorem 4) has shown that

\[
p \mid \binom{n}{k}
\]

for \( 0 \leq k \leq n \) if and only if \( n = a p^s - 1 \) with \( 1 \leq a < p \) and \( s \geq 0 \). Now suppose we have two primes \( p_1 \) and \( p_2 \) with \( p_1 < p_2 \leq n + 1 \) and

\[
p_1 \mid \binom{n}{k}
\]

for \( 0 \leq k \leq n \). By Fine's result, we have

\[
n + 1 = a_1 p_1^{s_1} = a_2 p_2^{s_2}
\]

with \( 1 \leq a_1 < p_1 \) and \( 1 \leq a_2 < p_2 \). But \( a_1 < p_1 < p_2 \) implies that \( p_2 \mid a_1 p_1^{s_1} \), so \( s_2 = 0 \) and \( n + 1 = a_2 < p_2 \), contrary to \( p_2 \leq n + 1 \). Hence there is at most one prime \( p \leq n + 1 \) such that

\[
p \mid \binom{n}{k}
\]

for \( 0 \leq k \leq n \) and such a \( p \) exists if and only if \( n + 1 = a p^s \) with \( 1 \leq a < p \). (More discussion related to this may be found in my survey paper: "Divisibility of Binomial and Multinomial Coefficients by Primes and Prime Powers," (to appear).)

By carefully examining the role of \( n + 1 \), we can deduce the following formulas for \( M(n) \).
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\[
M(n) = \begin{cases} 
\pi(n + 1) & \text{if } n + 1 \neq a^p \text{ with } 1 \leq a < p \\
\pi(n + 1) - 1 & \text{otherwise.}
\end{cases}
\]

\[
M(n) = \begin{cases} 
\pi(n) & \text{if } n + 1 \neq a^p \text{ with } 1 \leq a < p \leq n \\
\pi(n) - 1 & \text{otherwise.}
\end{cases}
\]

Hence \( M(n) \sim \pi(n) \sim [(\log n)/n]^{-1} \).

Incidentally, \( M(13) = M(15) = 5 \), contrary to what was asserted in the statement of the problem. The first place where \( M(n) > M(n + 1) \) is \( n = 83 \), where \( M(83) = 23 \) and \( M(84) = 22 \). The next cases are \( n = 89 \) and \( n = 104 \). From the expression for \( M(n) \), we have the following necessary conditions for such an \( n \): \( n + 1 \) must have three distinct prime factors and \( n + 2 \) must not be prime.

Also solved by the Proposer.

DISPLAY CASE

H-201 Proposed by Verner E. Hoggatt, Jr., San Jose State University, San Jose, California.

Copy 1, 1, 3, 8, \( \cdots \), \( F_{2n-2} \) \( (n \geq 1) \) down in staggered columns as in display C:

\[
\begin{array}{ccccccc}
1 \\
1 & 1 \\
3 & 1 & 1 \\
8 & 3 & 1 & 1 \\
21 & 8 & 3 & 1 & 1 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots
\end{array}
\]

(i) Show that the row sums are \( F_{2n+1} \) \( (n = 0, 1, 2, \cdots) \).

(ii) Show that, if the columns are multiplied by 1, 2, 3, \( \cdots \), sequentially to the right, then the row sums are \( F_{2n+2} \) \( (n = 0, 1, 2, \cdots) \).

(iii) Show that the rising diagonal sums \( (\gamma) \) are \( F_{n+1}^2 \) \( (n = 0, 1, 2, \cdots) \).

Solution by P. Bruckman, University of Illinois at Chicago, Chicago, Illinois.

(i) Let \( R_n \) denote the row sum of the \( (n + 1) \)th row, \( (n = 0, 1, 2, \cdots) \), with \( R_0 = 1 \).

\[
R_n = 1 + \sum_{k=0}^{n-1} F_{2n-2k} = 1 + \sum_{k=1}^{n} F_{2k} = 1 + \sum_{k=1}^{n} (F_{2k+1} - F_{2k-1}) = 1 + F_{2n+1} - 1 = F_{2n+1}.
\]
as asserted.

(ii) Let \( S_n \) denote the sum as defined in the problem, for the \((n + 1)^{th}\) row, \((n = 0, 1, 2, \cdots)\), with \( S_0 = 1 \). Then, if \( n \geq 1 \),

\[
S_n = \sum_{k=1}^{n} k F_{2n+2-2k} + (n + 1) = \sum_{k=0}^{n-1} (n - k)F_{2k+2} + (n + 1) = (n + 1) + \sum_{k=0}^{n-1} F_{2k+2} \sum_{i=0}^{n-k-1} 1
\]

\[
= (n + 1) + \sum_{i=0}^{n-1} \sum_{k=0}^{n-i-1} F_{2k+2} = (n + 1) + \sum_{i=0}^{n-1} (F_{2n-2i+1} - 1) = (n + 1) + \sum_{i=1}^{n} F_{2i+1} - n
\]

\[
= 1 + \sum_{i=1}^{n} (F_{2i+2} - F_{2i}) = 1 + F_{2n+2} - F_2 = F_{2n+2}
\]

as asserted. This is also true for \( n = 0 \).

(iii) Let \( T_n \) denote the rising diagonal sums. Then, if \( n \geq 2 \),

\[
T_n = \sum_{k=1}^{\frac{n}{2}} F_{4k} + 1, \text{ if } n \text{ is even; } T_n = \sum_{k=1}^{\frac{n-1}{2}} F_{4k-2}, \text{ if } n \text{ is odd; } T_0 = T_1 = 1.
\]

\[
T_{2m} = \sum_{k=1}^{m} F_{4k} + 1 = F_1 + \sum_{k=1}^{m} (F_{4k+1} - F_{4k-1}) = \sum_{i=0}^{2m} (-1)^i F_{2i+1};
\]

also,

\[
T_{2m+1} = \sum_{k=1}^{m+1} F_{4k-2} = \sum_{k=1}^{m+1} (F_{4k-1} - F_{4k-3}) = \sum_{i=0}^{2m+1} (-1)^{i+1} F_{2i+1}.
\]

Combining these results, we have

\[
T_n = \sum_{i=0}^{n} (-1)^{n-i} F_{2i+1} = \sum_{i=0}^{n} (-1)^{n-i} (F_{2i+1}^2 + F_i^2)
\]

\[
= \sum_{i=0}^{n} (-1)^{n-i} F_{2i+1}^2 - (-1)^{n-i+1} F_i^2 = (-1)^n F_{n+1}^2 - (-1)^{n+1} F_{n+1}^2 = 0 = F_{n+1}^2.
\]

This last result is also true for \( n = 0 \) and \( n = 1 \).

*Also solved by the Proposer and one unsigned solver.*