A COMBINATORIAL IDENTITY

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Define

$$f(n,k) = 2^n \sum_{i=k}^{c} (-1)^i \binom{n-i}{i}\binom{i}{k} 2^{-2i} , \tag{1}$$

where

$$c = \begin{cases} n/2, & n \text{ even} \\ (n-1)/2, & n \text{ odd} \end{cases} .$$

By induction, it is proved that

$$f(n,k) = (-1)^k \binom{n+1}{2k+1} = (-1)^k \binom{n+1}{n-2k} \quad \text{for} \quad 0 \le k \le c . \tag{2}$$

The usual induction procedure must be modified since the identity involves both n and k but only restricted values of k associated with each n. Figure 1 illustrates how the induction proceeds. For the n and k shown, the identity is valid at the darkened grid points. The letter label on a grid point or on an arrow refers to part A, B, C, or D of the proof.

Part A of the proof shows that when n even, assuming (2) is true for (n,k), $(n-1,k)$, and $(n-1, k-1)$, then (2) is true for $(n+1, k)$. This applies to all k associated with n and $n+1$ except for $k = 0$ and $k = n/2$. Part B shows that for n even, $k \ne 0$, $k \ne (n+2)/2$, assuming as in A that (2) is true for (n,k), adding the assumption that (2) is true for $(n, k-1)$, and using the result of A that (2) is true for $(n+1, k)$, then (2) is true for $(n+2, k)$. Part C shows that (2) is true for $(n,0)$ and Part D deals with the special cases of $(n, n/2)$ and $(n+1, n/2)$ for n even.

A. Starting with

$$\binom{n+1-i}{i}\binom{i}{k} \equiv \binom{n-i}{i}\binom{i}{k} + \binom{n-i}{i-1}\binom{i-1}{k} + \binom{n-i}{i-1}\binom{i-1}{k-1} \tag{3}$$

for $1 \le k \le i-1$, $i \le n/2$, n even, a factor of $(-1)^i 2^{n-2i}$ is introduced into each term. Each term in the equation is summed over $i = k+1, \cdots, n/2$. For notational convenience, call the result

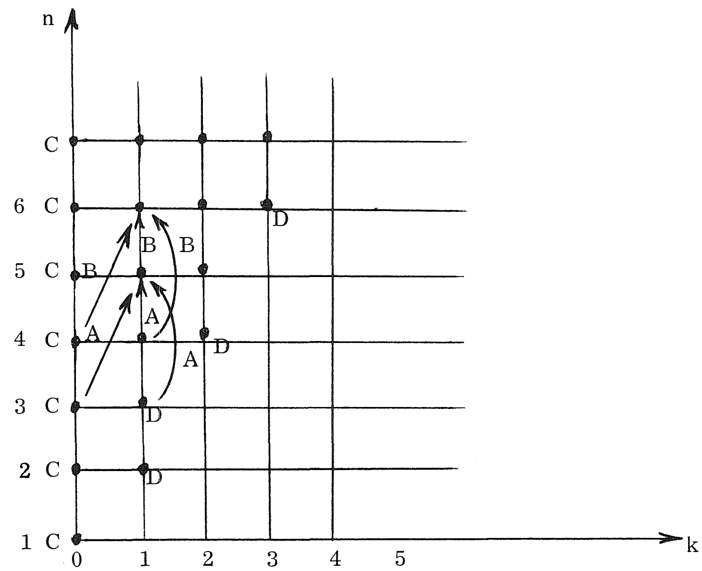

Figure 1

$$S_{1,n} = S_{2,n} + S_{3,n} + S_{4,n} \; . \tag{4}$$

It is found that, for n even,

$$\begin{aligned} S_{1,n} &= \left[f(n+1,\,k) - 2^{n+1-2k}(-1)^k \binom{n+1-k}{k}\right]\Big/2 \\ S_{2,n} &= f(n,k) - 2^{n-2k}(-1)^k \binom{n-k}{k} \\ S_{3,n} &= -f(n-1,\,k)/2 \\ S_{4,n} &= -\left[f(n-1,\,k-1) + 2^{n+1-2k}(-1)^k \binom{n-k}{k-1}\right]\Big/2 \; . \end{aligned} \tag{5}$$

If (2) is true for (n,k), $(n-1,\,k)$, and $(n-1,\,k-1)$, (4) can be solved for $f(n+1,\,k)$ and

$$f(n+1,\,k) = (-1)^k \binom{n+2}{n+1-2k} \tag{6}$$

for $1 \le k \le (n-2)/2$, n even.

B. Using (3) modified such that each n is replaced by $n+1$, a factor of

$$(-1)^i \, 2^{n+1-2i}$$

is introduced into each term and each term of the equation is summed over $i = k+1, \cdots, n/2$. The result is

$$\left[S_{1,n+1} - (1/2)(-1)^{\frac{n+2}{2}}\binom{\frac{n+2}{2}}{k}\right]$$

$$(4') \qquad = S_{2,n+1} + \left[S_{3,n+1} - (1/2)(-1)^{\frac{n+2}{2}}\binom{\frac{n}{2}}{k}\right] + \left[S_{4,n+1} - (1/2)(-1)^{\frac{n+2}{2}}\binom{\frac{n}{2}}{k-1}\right] .$$

If (2) is true for (n,k), $(n, k-1)$, and $(n+1, k)$, (4') can be solved for $f(n+2, k)$ and

$$(5') \qquad f(n+2, k) = (-1)^k\binom{n+3}{n+2-2k}$$

for $1 \le k \le n/2$, n even.

C. When $k = 0$, (3) reduces to the familiar identity

$$(3'') \qquad \binom{n+1-i}{i} \equiv \binom{n-i}{i} + \binom{n-i}{i-1}$$

for $1 \le i \le n/2$, n even, and (4) reduces to

$$(4'') \qquad S_{1,n} = S_{2,n} + S_{3,n} ,$$

where $S_{1,n}$, $S_{2,n}$, $S_{3,n}$ are as defined in (5).

Hence, if $f(n,0) = n+1$ and $f(n-1, 0) = n$, then $f(n+1, 0) = n+2$ for n even.

Similar modification of Part B leads to $f(n+2, 0) = n+3$ if $f(n,0) = n+1$ and $f(n+1, 0) = n+2$ for n even. Verifying by substitution into (1) that $f(2,0) = 3$ and $f(1,0) = 2$ completes the case of $k = 0$.

D. Finally by substitution into (1), it is verified that (2) is true for $(n, n/2)$ and $(n+1, n/2)$ for n even.