A COMBINATORIAL IDENTITY

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Define

\[ f(n, k) = 2^n \sum_{i=k}^{c} (-1)^i \binom{n-1}{i} \binom{i}{k} 2^{-2i}, \]

where

\[ c = \begin{cases} n/2, & \text{n even} \\ (n - 1)/2, & \text{n odd} \end{cases} \]

By induction, it is proved that

\[ f(n, k) = (-1)^k \binom{n + 1}{2k + 1} = (-1)^k \binom{n + 1}{n - 2k} \text{ for } 0 \leq k \leq c. \]

The usual induction procedure must be modified since the identity involves both \( n \) and \( k \) but only restricted values of \( k \) associated with each \( n \). Figure 1 illustrates how the induction proceeds. For the \( n \) and \( k \) shown, the identity is valid at the darkened grid points. The letter label on a grid point or on an arrow refers to part A, B, C, or D of the proof.

Part A of the proof shows that when \( n \) even, assuming (2) is true for \( (n, k), (n - 1, k), \) and \( (n - 1, k - 1) \), then (2) is true for \( (n + 1, k) \). This applies to all \( k \) associated with \( n \) and \( n + 1 \) except for \( k = 0 \) and \( k = n/2 \). Part B shows that for \( n \) even, \( k \neq 0, k \neq (n + 2)/2 \), assuming as in A that (2) is true for \( (n, k) \), adding the assumption that (2) is true for \( (n, k - 1) \), and using the result of A that (2) is true for \( (n + 1, k) \), then (2) is true for \( (n + 2, k) \). Part C shows that (2) is true for \( (n, 0) \) and Part D deals with the special cases of \( (n, n/2) \) and \( (n + 1, n/2) \) for \( n \) even.

A. Starting with

\[ \binom{n + 1 - i}{i} \binom{i}{k} = \binom{n - i}{i} \binom{i}{k} + \binom{n - i}{i - 1} \binom{i - 1}{k} + \binom{n - i}{i - 1} \binom{i - 1}{k - 1} \]

for \( 1 \leq k \leq i - 1, i \leq n/2, n \) even, a factor of \( (-1)^i 2^{n-2i} \) is introduced into each term. Each term in the equation is summed over \( i = k + 1, \ldots, n/2 \). For notational convenience, call the result
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\[ S_{1,n} = S_{2,n} + S_{3,n} + S_{4,n} \]

It is found that, for \( n \) even,

\[ S_{1,n} = \left[ f(n, k) - 2^{n+1-2k}(-1)^k \binom{n+1}{k} \right] / 2 \]
\[ S_{2,n} = f(n, k) - 2^{n-2k}(-1)^k \binom{n-k}{k} \]
\[ S_{3,n} = -f(n-1, k)/2 \]
\[ S_{4,n} = -\left[ f(n-1, k-1) + 2^{n+1-2k}(-1)^k \binom{n-k}{k-1} \right] / 2 \]

If (2) is true for \( (n, k), (n-1, k), \) and \( (n-1, k-1), (4) \) can be solved for \( f(n+1, k) \) and

\[ f(n+1, k) = (-1)^k \binom{n+2}{n+1-2k} \]

for \( 1 \leq k \leq (n-2)/2, n \) even.

B. Using (3) modified such that each \( n \) is replaced by \( n+1 \), a factor of

\[ (-1)^i 2^{n+1-2i} \]
is introduced into each term and each term of the equation is summed over \( i = k + 1, \ldots, n/2 \). The result is

\[
\left[ S_{1,n+1} - \frac{n+2}{2} \binom{n+2}{k} \right]
\]

(4')

\[
= S_{2,n+1} + \left[ S_{3,n+1} - \frac{n+2}{2} \binom{n}{k} \right]
\]

\[
+ \left[ S_{4,n+1} - \frac{n+2}{2} \binom{n}{k-1} \right].
\]

If (2) is true for \((n,k), (n, k - 1), \text{ and } (n + 1, k)\), (4') can be solved for \( f(n+2, k) \) and

\[
(5') 
\]

\[
f(n + 2, k) = (-1)^k \binom{n + 3}{n + 2 - 2k}
\]

for \( 1 \leq k \leq n/2, \ n \text{ even.} \)

C. When \( k = 0 \), (3) reduces to the familiar identity

\[
(3'') 
\]

\[
\binom{n + 1}{i - 1} = \binom{n - i}{i} + \binom{n - i}{i - 1}
\]

for \( 1 \leq i \leq n/2, \ n \text{ even,} \) and (4) reduces to

\[
(4'') 
\]

\[
S_{1,n} = S_{2,n} + S_{3,n},
\]

where \( S_{1,n}, S_{2,n}, S_{3,n} \) are as defined in (5).

Hence, if \( f(n,0) = n + 1 \) and \( f(n - 1, 0) = n \), then \( f(n + 1, 0) = n + 2 \) for \( n \text{ even.} \)

Similar modification of Part B leads to \( f(n + 2, 0) = n + 3 \) if \( f(n,0) = n + 1 \) and \( f(n + 1, 0) = n + 2 \) for \( n \text{ even.} \) Verifying by substitution into (1) that \( f(2,0) = 3 \) and \( f(1,0) = 2 \) completes the case of \( k = 0. \)

D. Finally by substitution into (1), it is verified that (2) is true for \((n, n/2)\) and \((n + 1, n/2)\) for \( n \text{ even.} \)