FUNCTIONAL EQUATIONS WITH PRIME ROOTS FROM ARITHMETIC EXPRESSIONS FOR $\mathscr{G}_\alpha$

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1. In this article, a generalized form of Euler's law concerning the sigma function will be obtained and used to derive expressions for $\mathscr{G}_\alpha$ which contain just functions involving addition and multiplication. These will be substituted in the equations

$$\mathscr{G}_\alpha(n) - n^\alpha - 1 = 0 \tag{1}$$

to obtain equations with classes of solutions identical with the class of prime numbers.

2. Let

$$F(n) = \sum_{d|n} f(d) .$$

<u>Proposition 1.</u> If

$$\sum_{n=1}^{\infty} F(n)\, x^n$$

converges on some interval about 0, then

$$0 = nR(n) + \sum_{a=1}^{n} F(a)R(n-a) , \tag{2}$$

where

$$\sum_{n=0}^{\infty} R(n)\, x^n = \prod_{n=1}^{\infty} (1 - x^n)^{f(n)/n} . \tag{3}$$

The proof mimics Euler's for the case f = identity, which is the recursive expression for sum of divisors he obtained by describing R. [1]

Proof.

$$\sum_{n=1}^{\infty} f(n)\, x^n/(1 - x^n) = \sum_n f(n) \sum_k x^{nk}$$

$$\begin{aligned} = f(1)x &+ f(1)x^2 + f(1)x^3 + f(1)x^4 + f(1)x^5 + f(1)x^6 + \cdots \\ &+ f(2)x^2 \qquad\quad + f(2)x^4 \qquad\quad + f(2)x^6 + \cdots \\ &\qquad\qquad + f(3)x^3 \qquad\qquad\quad\;\; + f(3)x^6 + \cdots \\ &\qquad\qquad\qquad\qquad + f(4)\, x^4 + \cdots \\ &\qquad\qquad\qquad\qquad\qquad + f(5)x^5 + \cdots \\ &\qquad\qquad\qquad\qquad\qquad\qquad + f(6)x^6 + \cdots \end{aligned}$$

$$= \sum_{n=1}^{\infty} x^n \sum_{d|n} f(d) = \sum_{n=1}^{\infty} F(n)\, x^n .$$

That is,

$$\sum_{n=1}^{\infty} f(n)\, x^n/(1 - x^n) = \sum_{n=1}^{\infty} F(n)\, x^n . \tag{4}$$

Suppose

$$0 < \Pi\,(1 - x^n)^{f(n)/n} < \infty \tag{5}$$

on some interval about 0. We show that (2) holds under (5) and then that (5) holds when

$$\sum_{n=1}^{\infty} F(n)\, x^n$$

converges on some interval about 0.

Let (5) hold. We have the identity:

$$\log \prod_{n=1}^{\infty} (1 - x^n)^{f(n)/n} = \sum_{1}^{\infty} f(n)/n \log (1 - x^n) .$$

Differentiating, and substituting from (3) as (5) permits:

$$\sum_1^\infty -f(n)x^{n-1}/(1 - x^n) = \frac{\dfrac{d}{dx}\left[\prod_1^\infty (1 - x^n)^{f(n)/n}\right]}{\prod_1^\infty (1 - x^n)^{f(n)/n}}$$

$$= \left(\frac{d}{dx}\sum_0^\infty R(m)x^m\right) \Big/ \sum_0^\infty R(m)x^m$$

$$= \sum_0^\infty mR(m)x^{m-1} \Big/ \sum_0^\infty R(m)x^m .$$

Hence, by (4),

$$-\sum_0^\infty mR(m)x^m \Big/ \sum_0^\infty R(m)x^m = \sum_1^\infty f(n)x^n / (1 - x^n) = \sum_1^\infty F(n)x^n . \tag{6}$$

and Eq. (6) gives:

$$0 = \left(\sum_1^\infty F(n)x^n\right)\left(\sum_0^\infty R(m)x^m\right) + \sum_0^\infty mR(m)x^m .$$

So, for each $n \geq 0$, the coefficient x^n is 0:

$$0 = \sum_{a=1}^n F(a)R(n - a) + nR(n) .$$

It remains to show that (5) holds when

$$\sum_{n=1}^\infty F(n)\,x^n$$

converges on some interval about 0. By Eq. (6),

$$\sum_1^\infty F(n)x^n = -xd/dx \log P(x) ,$$

where

$$P(x) = \prod_{1}^{\infty} (1 - x^n)^{f(n)/n} .$$

Therefore,

$$P(x) = \exp \int - \sum_{1}^{\infty} F(n)x^{n-1}\,dx . \tag{7}$$

Hence $P(x) = 0$ iff

$$\int \sum_{1}^{\infty} F(n)x^{n-1}dx = \infty$$

iff

$$\sum_{1}^{\infty} \frac{F(n)x^n}{n} = \infty$$

and $P(x) = \infty$ iff

$$\sum_{1}^{\infty} (1/n)F(n)x^n = -\infty .$$

Thus (5) holds iff

$$\left| \sum_{1}^{\infty} (1/n)F(n)x^n \right| < \infty$$

on some interval about 0, and this is the case when

$$\left| \sum_{1}^{\infty} F(n)\,\kappa^n \right| < \infty$$

on the same interval. Q.E.D.

Now it is necessary to show that the conditions of Proposition 1 apply to $\mathcal{G}_\alpha$. Actually, we show a little more.

<u>Proposition 2</u>. Let

$$\sum_{d|n} f(d) = F(n) .$$

Then,

$$\left| \sum F(n)\, x^n \right| < \infty$$

on some interval about 0 if and only if

$$\left|\sum f(n)x^n\right| < \infty$$

on some interval about 0.

Proof.

$$\left|\sum F(n)x^n\right| < \infty \to \left|\sum f(n)x^n/1 - x^n\right| < \infty \to \left|\sum f(n)x^n\right| < \infty$$

by (4) and comparison. For the other direction, let

$$\left|\sum f(n)x^n\right| < \infty \quad .$$

By the root test,

$$\lim\sup |f(n)|^{\frac{1}{n}} < \infty \ .$$

That is, $\sup L_i < \infty$, where

$$L_i = \lim_k |f(a_{ik})|^{\frac{1}{a_{ik}}}$$

on some sequence $\{a_{ik}\}$.

Define $\{c_k\}$ by:

$$|f(c_k)| = \max_{d|a_k} |f(d)|$$

for a sequence $\{a_k\}$. For each k, c_k is one of the divisors of a_k. Then,

$$\lim |f(c_k)|^{\frac{1}{c_k}} \le \sup L_i < \infty,$$

and over all sequences $\{a_k\}$ the $\{c_k\}$ are bounded by:

$$\sup_{\{a_k\}} \lim |f(c_k)|^{\frac{1}{c_k}} \le \sup L_i < \infty \quad .$$

That is,

$$\sup_{\{a_k\}} \lim \left[\max_{d|a_k} |f(d)|\right]^{\frac{1}{c_k}} \le \sup L_i \ .$$

Now

$$\left[\max_{d|a_k} |f(d)|\right]^{\frac{1}{a_k}} \le \left[\max_{d|a_k} |f(d)|\right]^{\frac{1}{c_k}} .$$

So:

$$\sup_{\{a_k\}} \lim \left[\max_{d|a_k} |f(d)|\right]^{\frac{1}{a_k}} \le \sup L_i < \infty .$$

That is,

$$\limsup \max_{d|n} |f(d)|^{\frac{1}{n}} \le \sup L_i < \infty .$$

Now, we demonstrate below that

$$\left|\sum \tau(n)\, x^n\right| < \infty$$

on some interval about 0, where τ is the number-of-divisors function. The demonstration below is valid but clearly circuitous. Thus,

$$\limsup |\tau(n)|^{\frac{1}{n}} < \infty$$

by the root test, and

$$\limsup \left[\tau(n) \max_{d|n} |f(d)|\right]^{\frac{1}{n}} = \limsup |\tau(n)|^{\frac{1}{n}} \left[\max_{d|n} |f(d)|\right]^{-\frac{1}{n}}$$

$$\le \limsup |\tau(n)|^{\frac{1}{n}} \limsup \left[\max_{d|n} |f(d)|\right]^{\frac{1}{n}} < \infty .$$

Thus,

$$\sum_n x^n \tau(n) \max_{d|n} |f(d)| < \infty$$

on some interval about 0. Then,

$$\left|\sum F(n)x^n\right| \le \sum |F(n)|x^n \le \sum \sum_{d|n} |f(d)|x^n$$

$$\le \sum \tau(n) \max_{d|n} |f(d)|\, x^n < \infty$$

q. e. d.

We repair the gap in the proof of Proposition 2, the assertion without demonstration that

$$\sum \tau(n)\, x^n$$

converges on some interval about 0, by comparing this sum with another. The result is obvious on comparing $\tau(n)$ with the identity function:

$$\left|\sum nx^n\right| < \infty$$

on (-1, 1).

One more proposition is needed to finish the background for a demonstration that Proposition 1 applies to $\mathcal{G}_\alpha$.

Proposition 3:

$$\sum 1/n\ F(n)\,x^n$$

converges on some interval about 0 iff

$$\sum F(n)\,x^n$$

converges on some interval about 0.

Proof. Under the hypothesis that

$$\sum 1/n\ F(n)\,x^n$$

converges we have by the root test:

$$\limsup \left|F(n)(1/n)\right|^{\frac{1}{n}} < \infty .$$

That is,

$$\sup_{\{a_k\}} \lim \left|F(a_k)(1/a_k)\right|^{\frac{1}{a_k}} < \infty .$$

Now, clearly when

$$\left|F(a_k)(1/a_k)\right|^{\frac{1}{a_k}}$$

converges, its limit is

$$\lim \left|F(a_k)\right|^{\frac{1}{a_k}}$$

Also, it is clear that

$$\left|F(a_k)\right|^{\frac{1}{a_k}}$$

converges if and only if

$$\left|F(a_k)(1/a_k)\right|^{\frac{1}{a_k}} ,$$

too, converges. So

$$\limsup \left|F(n)\right|^{\frac{1}{n}} = \sup_{\{a_k\}} \lim \left|F(a_k)\right|^{\frac{1}{a_k}} = \sup_{\{a_k\}} \lim \left|F(a_k)(1/a_k)\right|^{\frac{1}{a_k}}$$
$$= \limsup \left|F(n)(1/n)\right|^{\frac{1}{n}} .$$

So
$$\sum F(n)\, x^n$$
converges on some interval about 0. The other direction is similar, or by comparison, q. e. d.

Now we prove that Proposition 1 may be applied to $\mathcal{G}_\alpha$.

Proposition 4.
$$\sum \mathcal{G}_\alpha(n)\, x^n$$
converges on some interval about 0.

Proof. $\sum x^n$ converges on $[0,1)$. Apply Proposition 3 inductively: for each α,
$$\sum n^\alpha x^n$$
converges on some interval. Then, by Proposition 2,
$$\sum \mathcal{G}_\alpha(n)\, x^n$$
converges. q. e. d. Proposition 1 now yields a recursive relation on $\mathcal{G}_\alpha$ in terms of the coefficients of the power series for $P(x)$ with $f(n) = n^\alpha$. $P(x)$ is an infinite product and, in order to determine an expression for $\mathcal{G}_\alpha$ which is recursive in addition and multiplication, we express the coefficients of the power series for $P(x)$ as the coefficients of the expansion of a finite product.

Proposition 5.
$$0 = nR(n) + \sum_{a=1}^{n} \mathcal{G}_\alpha(a) R(n-a) ,$$
where $R(k)$ = coefficient of x^k in
$$\prod_{n=1}^{k} (1 - x^n)^{n^{\alpha-1}} .$$

Proof. Applying Proposition 1, to
$$\prod_{n=1}^{\infty} (1 - x^n)^{n^{\alpha-1}} = \sum_{n=0}^{\infty} S(n) x^n :$$
Let
$$\prod_{1}^{k} (1 - x^n)^{n^{\alpha-1}} = \sum_n \overline{R}_k(n) x^n$$
(Definition). Then
$$\sum_n \overline{R}_{k+1}(n)\, x^n = \prod_{n=1}^{k+1} (1 - x^n)^{n^{\alpha-1}} = (1 - x^{k+1})^{[k+1]^{\alpha-1}} \times \sum_n \overline{R}_k(n) x^n$$
$$= \sum_{r=0}^{[k+1]^{\alpha-1}} \binom{[k+1]^{\alpha-1}}{r} (-1)^r x^{(k+1)r} \times \sum_n \overline{R}_k(n) x^n =$$

$$= \sum_n \overline{R}_k(n)\, x^n + \sum_{r=1}^{[k+1]^{\alpha-1}} \binom{[k+1]^{\alpha-1}}{r} (-1)^r x^{(k+1)r} x \sum_n \overline{R}_k(n)\, x^n$$

$$= \sum_n \overline{R}_{k+1}(n)\, x^n \ .$$

None of the terms in the second summand have exponents $\leq k$. Thus

$$\overline{R}_k(i) = \overline{R}_{k+1}(i)$$

for all $i \leq k$. Indeed, $\overline{R}_k(i) = \overline{R}_1(i)$ for all i and 1 such that $i \leq k \leq 1$. Thus

$$\sum S(n)\, x^n = \lim_k \prod_1^k (1 - x^n)^{n^{\alpha-1}} = \lim_k \sum_n \overline{R}_k(n)\, x^n = \sum_n \lim_k \overline{R}_k(n)\, x^n = \sum_n \overline{R}_n(n)\, x^n,$$

and $\overline{R}_n(n) = S(n)$. q.e.d.

It is now possible to define a function, which turns out to be $\mathcal{G}_\alpha$, which is expressible in terms of just addition and multiplication, and which leads to the equation mentioned in the title.

Define $F_\alpha(1) = 1$ and, supposing F_α defined on $1, 2, \cdots, n-1$, let $F_\alpha(n)$ satisfy

$$0 = nR(n) + \sum_{a=1}^{n} F_\alpha(a) R(n-a) \ ,$$

where R is defined as in the statement of Proposition 5. Then, by Proposition 5, $F_\alpha = \mathcal{G}_\alpha$, and F_α satisfies $0 = F_\alpha(n) - n^\alpha - 1$ just when $\underline{n}$ is a prime number.

REFERENCE

1. Euler, Opera Omnia, Series 1, Vol. 2, pp. 241-253, "Discovery of a Most Extraordinary Law of the Numbers Concerning the Sum of Their Divisors."