SPANNING TREES AND FIBONACCI AND LUCAS NUMBERS

A. J. W. HILTON
The University of Reading, Whiteknights, Reading, England

1. INTRODUCTION

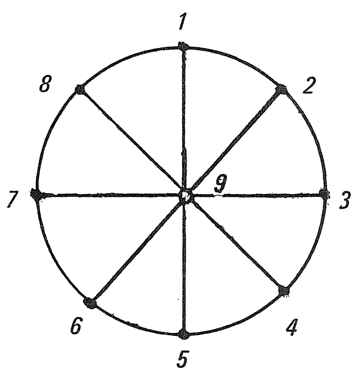
The Fibonacci numbers F_n are defined by

$$F_1 = F_2 = 1, \quad F_{n+2} = F_{n+1} + F_n \quad (n \geqslant 1),$$

and the Lucas numbers L_n by

$$L_1 = 1, \quad L_2 = 3, \quad L_{n+2} = L_{n+1} + L_n \quad (n \geqslant 1).$$

We shall use the graph theoretic terminology of Harary [2]. A *wheel* on $n + 1$ points is obtained from a cycle on n points by joining each of these n points to a further point. This cycle is known as the *rim* of the wheel, the other edges are the *spokes*, and the further point is the *hub*. A *fan* is what is obtained when one edge is removed from the rim of a wheel. We also refer to the rim and the spokes of a fan, but use the word *pivot* instead of hub. We give now an illustration of a labelled wheel and a labelled fan on 9 points.



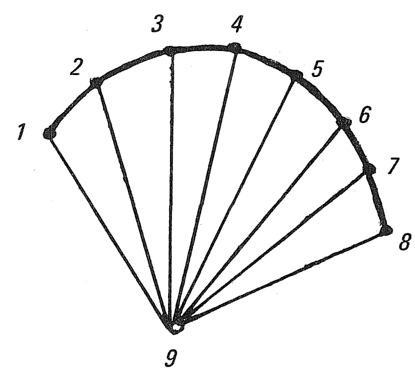


Figure 1

A *composition* of the positive integer n is a vector $(a_1, a_2, \cdots, a_k)$ whose components are positive integers such that $a_1 + a_2 + \cdots + a_k = n$. If the vector has order k then the composition is a k-part composition.

For $n \geqslant 2$ the number of spanning trees of a labelled wheel on $n + 1$ points is $L_{2n} - 2$, and the number of spanning trees of a labelled fan on $n + 1$ points is F_{2n}. References concerning the first of these results may be found in [3]; both results are proved simply in [4].

In this paper, by simple new combinatorial arguments, we derive both old and new formulae for the Fibonacci and Lucas numbers.

2. A SIMPLE COMBINATORIAL PROOF THAT $F_{2n+2m} = F_{2n+1}F_{2m} + F_{2n}F_{2m-1}$

Let the number of spanning trees of a labelled fan on $n+1$ points be f_n, and the number of those spanning trees

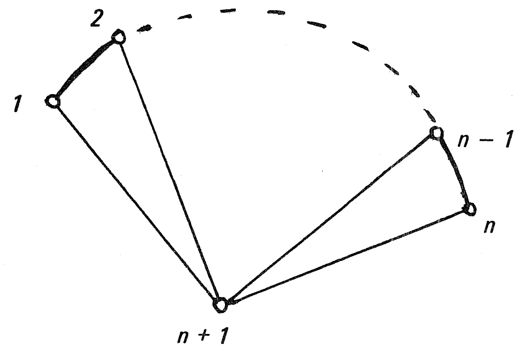


Figure 2

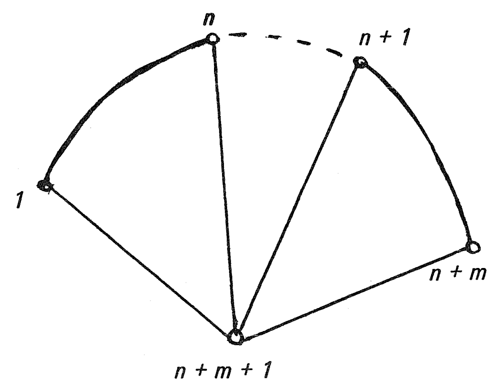


Figure 3

of a labelled fan on $n+1$ points which include a specified leading edge ($\{1, n+1\}$ in Fig. 2) be e_n. Clearly

(1) $$e_{n+1} = e_n + f_n \qquad (n \geq 1).$$

Now consider a fan on $n+m+1$ points. This may be thought of as two fans A and B, connected at the pivot and at two points labelled n and $n+1$ as indicated in Fig. 3. Then

(2) $$f_{n+m} = f_n f_m + f_n e_m + e_n f_m \qquad (n, m \geq 1)$$

so

(3) $$f_{n+m} = e_{n+1} f_m + f_n e_m \qquad (n, m \geq 1)$$

by (1). In formula (2) $f_n f_m$ is the number of those spanning trees which do not include $\{n, n+1\}$. The restrictions of a spanning tree which includes $\{n, n+1\}$ to A and to B are either a spanning tree of A and a spanning subgraph of B consisting of two trees, one including $\{n+1\}$, the other including $\{n+m+1\}$, or are a spanning tree of B and a spanning subgraph of A consisting of two trees, one including $\{n\}$, the other including $\{n+m+1\}$. Therefore, the number of spanning trees which include $\{n, n+1\}$ is $f_n e_m + e_n f_m$. But $f_n = F_{2n}$, and it is shown in [4] that $e_n = F_{2n-1}$. Therefore, from (3),

$$F_{2n+2m} = F_{2n+1} F_{2m} + F_{2n} F_{2m-1} \qquad (n, m \geq 1).$$

The corresponding formula for L_{2n+2m} does not appear to come through so readily from this type of argument.

3. COMPOSITION FORMULAE FOR F_{2n}

If $(a_1, \cdots, a_k)$ is a composition of n, then the number of spanning trees of the fan in Fig. 2 which exclude

$$\{a_1, a_1+1\}, \{a_1+a_2, a_1+a_2+1\}, \cdots \{a_1+\cdots+a_{k-1}, a_1+\cdots+a_{k-1}+1\}$$

but include all other edges of the rim is $a_1 a_2 \cdots a_k$, for this is the number of different combinations of spokes which such a spanning three may include. Therefore

(4) $$F_{2n} = \sum_{\gamma(n)} a_1 a_2 \cdots a_k ,$$

where $\gamma(n)$ indicates summation over all compositions $(a_1 \cdots, a_k)$ of n, the number of components being variable. This formula is due to Moser and Whitney [6].

Hoggatt and Lind [5] have shown that this formula may be inverted to give

$$-n = \sum_{\gamma(n)} (-1)^k F_{2\alpha_1} F_{2\alpha_2} \cdots F_{2\alpha_k} .$$

This may be demonstrated combinatorially as follows. The number of spanning trees of the fan in Fig. 2 which do not have any rim edges missing is n. The total number of spanning trees is F_{2n}. For a given composition $(a_1, \cdots, a_k)$ of n with $k \geqslant 2$, the number of spanning trees which do not contain the edges $\{a_1, a_1 + 1\}$,

$$\{a_1 + a_2, a_1 + a_2 + 1\}, \cdots, \{a_1 + \cdots + a_{k-1}, a_1 + \cdots + a_{k-1} + 1\}$$

is $F_{2\alpha_1} F_{2\alpha_2} \cdots F_{2\alpha_k}$. Therefore, by the Principle of Inclusion and Exclusion (see Riordan [7], Chapter 3)

$$n = \sum_{\gamma(n)} (-1)^{k-1} F_{2\alpha_1} F_{2\alpha_2} \cdots F_{2\alpha_k} .$$

Of course it now follows that

$$F_{2n} = n + \sum_{k=2}^{n} \sum_{\gamma_k(n)} (-1)^k F_{2\alpha_1} F_{2\alpha_2} \cdots F_{2\alpha_k} , \tag{5}$$

where $\gamma_k(n)$ denotes summation over all k-part compositions of n.

4. COMPOSITION FORMULAE FOR $L_{2n} - 2$.

The formulae in this section are analogous to the formulae (4) and (5) of the previous section. The main difference is that the formulae in this section are obtained from the wheel in Fig. 4, whereas in the last section they were obtained from the fan in Fig. 2.

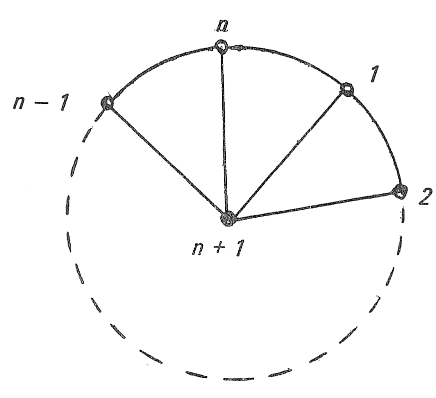


Figure 4

If $(a_1, \cdots, a_k)$ is a composition of n, and j is an integer, $0 \leqslant j < n$, then the number of spanning trees of the wheel in Fig. 4 which exclude the edges

$$\{a_1 + j, a_1 + j + 1\}, \{a_1 + a_2 + j, a_1 + a_2 + j + 1\}, \cdots, \{a_1 + \cdots + a_k + j, a_1 + \cdots + a_k + j + 1\}$$

[the integers here being taken modulo n], but include all the remaining edges in the rim, is $a_1 a_2 \cdots a_n$. If we sum over all such compositions into k parts and all possible values of j, we obtain

$$n \sum_{\gamma_k(n)} a_1 a_2 \cdots a_k .$$

But this sum counts each spanning tree which has exactly k specified edges on the rim excluded, precisely k times. Therefore the number of spanning trees which exclude exactly k edges of the rim is

$$\frac{n}{k} \sum_{\gamma_k(n)} a_1 a_2 \cdots a_n .$$

Therefore

$$L_{2n} - 2 = n \sum_{k=1}^{n} \frac{1}{k} \sum_{\gamma_k(n)} a_1 a_2 \cdots a_k .$$

i.e.,

$$L_{2n} - 2 = \sum_{\gamma(n)} \frac{n a_1 a_2 \cdots a_k}{k} ,$$

a formula which is analogous to (4).

We now find a formula for $L_{2n} - 2$ which is analogous to (5). The number of spanning trees of a wheel which do not have any rim edges missing is 0. The total number of spanning trees of a wheel is $L_{2n} - 2$. For a given composition $(a_1, a_2, \cdots, a_k)$ of n, and a given integer j, $0 \leqslant j < n$, the number of spanning trees which do not contain the edges

$$\{a_1 + j, a_1 + j + 1\}, \{a_1 + a_2 + j, a_1 + a_2 + j + 1\}, \cdots, \{a_1 + \cdots + a_k + j, a_1 + \cdots + a_k + j + 1\}$$

is $F_{2\alpha_1} F_{2\alpha_2} \cdots F_{2\alpha_k}$. By a similar argument to that just used above, the sum

$$\frac{n}{k} \sum_{\gamma_k(n)} F_{2\alpha_1} F_{2\alpha_2} \cdots F_{2\alpha_k}$$

is the sum taken over all combinations of k edges from the rim of the number of spanning trees which do not contain any of the k rim edges of the combination. Therefore, by the Principle of Inclusion and Exclusion

$$0 = L_{2n} - 2 + \sum_{k=1}^{n} (-1)^k \frac{n}{k} \sum_{\gamma_k(n)} F_{2\alpha_1} F_{2\alpha_2} \cdots F_{2\alpha_k} .$$

Therefore

$$L_{2n} - 2 = \sum_{\gamma(n)} (-1)^{k-1} \frac{n}{k} F_{2\alpha_1} F_{2\alpha_2} \cdots F_{2\alpha_k} ,$$

a formula which is analogous to (5).

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