A GENERALIZATION OF THE HILTON-FERN THEOREM ON THE EXPANSION OF FIBONACCI AND LUCAS NUMBERS

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1. INTRODUCTION

The object of this note is to generalize Hilton's extension [2] of Fern's theorem [1] to sequences of arbitrary order. Ferns found a general method by which products of Fibonacci and Lucas numbers of the form

$$u_{x_1}u_{x_2}\cdots u_{x_n}$$

could be expressed as a linear function of the u_n. Hilton extended Fern's results to include effectively the generalized sequence of numbers of Horadam [3].

We shall extend the result to linear recursive sequences of order r which satisfy the recurrence relation

$$(1.1)\qquad W_{s,n+r}^{(r)} = \sum_{j=1}^{r} (-1)^{j+1} P_{rj} W_{s,n+r-j}^{(r)} \qquad (s = 0, 1, \cdots, r-1;\ n \geq r)$$

where the P_{rj} are arbitrary integers, and for suitable initial values $W_{s,n}^{(r)}$, $n = 0, 1, \cdots, r-1$. When $r = 2$, we have Horadam's sequence. We are in effect supplying an elaboration of the results of Moser and Whitney [4] on weighted compositions.

Modifying Williams [5] let a_{rj} be the r distinct roots of the auxiliary equation

$$(1.2)\qquad x^r = \sum_{j=1}^{r} (-1)^{j+1} P_{rj} x^{r-j} \quad ,$$

where

$$(1.3)\qquad a_{rj} = \frac{1}{r} \sum_{k=0}^{r-1} W_{k,r-1}^{(r)}\, d^k w^{-jk} \qquad (j = 1, 2, \cdots, r)$$

in which d is the determinant of the Vandermonde matrix

$$\begin{bmatrix} 1 & 1 & \cdots & 1 \\ a_{r1} & a_{r2} & \cdots & a_{rr} \\ & \cdots & & \\ a_{r1}^{r-1} & a_{r2}^{r-1} & \cdots & a_{rr}^{r-1} \end{bmatrix}$$

and $w = \exp(2i\pi/r)$, $i^2 = -1$. (This is not as general as Williams' definition, but it is adequate for our present purpose.) When $r = 2$,

$$a_{2j} = \tfrac{1}{2}(W_{0,3}^{(2)} + (-1)^j dW_{1,3}^{(2)})$$

which agrees with Hilton.

We shall frequently use the fact that

$$\sum_{j=1}^{r} w^{-ij} = r\delta_{io},$$

where δ_{ij} is the Kronecker delta.

2. PRELIMINARY RESULTS

The first result we need is that

(2.1) $$W_{s,r+1}^{(r)} = d^{-s} \sum_{j=0}^{r-1} a_{rj} w^{sj} \qquad (s = 0, 1, \cdots, r-1).$$

Proof:

$$\sum_{j=0}^{r-1} a_{rj} w^{ij} = \frac{1}{r} \sum_{k=0}^{r-1} W_{k,r+1}^{(r)} d^k \sum_{j=0}^{r-1} w^{(i-k)j}$$
$$= \frac{1}{r} W_{i,r+1}^{(r)} d^i r ,$$

from which the result follows.

This suggests that we set

(2.2) $$W_{s,n+r}^{(r)} = d^{-s} \sum_{j=0}^{r-1} a_{rj}^n w^{sj} \qquad (s = 0, 1, \cdots, r-1),$$

and it remains to see whether the $W_{s,n}^{(r)}$ of formula (2.2) satisfy the recurrence relation (1.1).

The right-hand side of this recurrence relation is

$$\sum_{k=1}^{r} \sum_{m=0}^{r-1} (-1)^{k+1} d^{-s} a_{rm}^{n-k} w^{sm} P_{rk}$$

(from (2.2))

$$= d^{-s} \sum_{m=0}^{r-1} \left(\sum_{k=1}^{r} (-1)^{k+1} a_{rm}^{r-k} P_{rk} \right) a_{rm}^{n-r} w^{sm}$$

$$= d^{-s} \sum_{m=0}^{r-1} a_{rm}^r a_{rm}^{n-r} w^{sm}$$

(from (1.2))

$$= W_{s,n+r}^{(r)} \qquad \text{as required}$$

(from (2.2)). It follows then that

(2.3) $$a_{rj}^n = \frac{1}{r} \sum_{k=0}^{r-1} W_{k,n+r}^{(r)} d^k w^{-jk}$$
$$(j = 1, 2, \cdots, r).$$

Proof: From Eq. (2.2), we have that

$$\sum_{j=0}^{r-1} a_{rj}^n w^{ij} = \frac{1}{r} W_{i,r+1}^{(r)} d^i r$$
$$= \frac{1}{r} \sum_{k=0}^{r-1} W_{k,r+1}^{(r)} d^k \sum_{j=0}^{r-1} w^{(i-k)j}$$
$$= \sum_{j=0}^{r-1} \left(\frac{1}{r} \sum_{k=0}^{r-1} W_{k,r+1}^{(r)} d^k w^{-jk} \right) w^{ij}$$

from which we obtain the result.

3. HILTON-FERN THEOREM

Following Hilton let

(3.1)
$$S_m^n = \sum_{\Sigma k=m} \prod_{i=1}^{n} W_{k,x_i+r}^{(r)} \qquad (k = 0, 1, \cdots, r-1),$$

where we have all permutations of $(x_1, \cdots, x_n)$. For example, when $r = 2$, we get

$$S_0^n = \sum W_{0,x_1+2}^{(2)} W_{0,x_2+2}^{(2)} \cdots W_{0,x_{n-1}+2}^{(2)} W_{0,x_n+2}^{(2)},$$

and

$$S_1^n = \sum W_{0,x_1+2}^{(2)} W_{0,x_2+2}^{(2)} \cdots W_{0,x_{n-1}+2}^{(2)} W_{1,x_n+2}^{(2)},$$

and so on, as in Hilton.

Theorem: For S_m^n defined in formula (3.1),

$$W_{s,x_1+x_2+\cdots+x_n+r}^{(r)} = r^{-n} \sum_{j=0}^{r-1} \sum_{k=0}^{(r-1)n} (dw^{-j})^{k-s} S_k^n .$$

Proof: Let

$$X_n = \sum_{i=1}^{n} x_i .$$

Then

$$a_{rj}^{X_n} = \prod_{x_i=1}^{n} a_{rj}^{x_i} = \frac{1}{r^n} \prod_{x_i=1}^{n} \sum_{k=0}^{r-1} W_{k,x_i+r}^{(r)} d^k w^{-jk}$$

$$= r^{-n}(S_0^n + dw^{-j} S_1^n + \cdots + (dw^{-j})^{(r-1)n} S_{(r-1)n}^n)$$

$$= r^{-n} \sum_{k=0}^{(r-1)n} (dw^{-j})^k S_k^n .$$

Thus

$$W_{s,X_n+r}^{(r)} = d^{-s} \sum_{j=0}^{r-1} a_{rj}^{X_n} w^{sj}$$

(from (2.2))

$$= r^{-n} d^{-s} \sum_{j=0}^{r-1} \sum_{j=0}^{(r-1)n} d^k w^{(s-k)j} S_k^n ,$$

as required. For example,

$$W_{0,x_1+x_2+\cdots+x_n+2}^{(2)} = (½)^n \sum_{j=0}^{1} \sum_{k=0}^{n} (dw^{-j})^k S_k^n$$

$$= (½)^n \sum_{k=0}^{n} (d^k + (-d)^k) S_k^n$$

$$= \frac{1}{2^{n-1}} (S_0^n + d^2 S_2^n + \cdots),$$

and

$$W_{1,x_1+x_2+\cdots+x_n+2}^{(2)} = (½)^n \sum_{j=0}^{1} \sum_{k=0}^{n} (dw^{-j})^{k-1} S_k^n$$

$$= (½)^n \sum_{k=0}^{n} (d^{k-1} + (-d)^{k-1}) S_k^n = \frac{1}{2^{n-1} d} (dS_1^n + d^3 S_3^n + \cdots),$$

which agree with Hilton when his $A = B = 1$. These results could be made more general by generalizing the definition of a_{rj} along the lines of Williams.

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TO MARY ON OUR 34th ANNIVERSARY

HUGO NORDEN
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Our wedlock year is thirty-four,
A number Fibo did adore,
He'd say, "Your shape is really great,
A perfect one point six one eight."

As everyone around can see,
You're pure Dynamic Symmetry,
And when demurely you stroll by
All know you are exactly Phi.

Proportions are what makes things run,
Like eight, thirteen and twenty-one,
Then, next in line is thirty-four,
But, wait, there's still a whole lot more.

In nineteen hundred ninety-five
Our wedlock year is fifty-five,
There's much more living yet in store,
Today is only thirty-four!

So stay the way you are today,
Don't work too hard, take time to play,
And stay point six one eight to one
So we can still enjoy the fun.

Hugo

April 7, 1974